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This is part two of a two-part SMSG mathematics text for junior high school students. Key ideas emphasized are structure of arithmetic from an algebraic viewpoint, the real number system as a progressing development, and metric and non-metric relations in geometry. Chapter topics include real numbers, similar triangles, variation, non-metric polyhedrons, volumes and surface areas, relative error, permutations and combinations, and probability. Slight revisions are contained in a later edition. (MP)

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**SCHOOL
MATHEMATICS
STUDY GROUP**

MATHEMATICS FOR JUNIOR HIGH SCHOOL

VOLUME II (Part 2)

SE 027 982

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MATHEMATICS FOR JUNIOR HIGH SCHOOL

Volume II (Part 2)

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Mathematics for Junior High School

Volume II (Part 2)

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UNIT 5

REAL NUMBERS

In your study of mathematics you have used several number systems. You began with the counting numbers, and probably knew a good deal about these numbers before you entered the first grade in school. These numbers are so familiar that it is easy to overlook some of the ways in which the system of counting numbers differs from other systems. Consider the following questions:

(a) Think of a particular counting number. What is the next smaller counting number? the next larger? If n represents a counting number, what represents the next smaller counting number? the next larger?

(b) Is there a counting number which cannot be used as a replacement for n in your answer to question (a)? Why?

(c) Is there a smallest counting number? a largest? If so, what are they?

(d) Under what operations is the set of counting numbers closed? not closed? Give illustrations.

(e) What two operations are defined for the counting numbers? What two operations are defined as inverses of the first two?

(f) How many counting numbers are there between 8 and 11? between 3002 and 4002? between 168 and 169? Between any two counting numbers is there always another counting number?

This year you have studied a new set of numbers, the integers. The set of integers contains the set of counting numbers (called positive integers). The negative integers may be defined in this way: If a is a counting number, then $(-a)$ is a number such that

$a + (-a) = 0$. What integer is neither positive nor negative?

5-1. Rational Numbers

The set of integers is contained in another set of numbers which you have studied, the rational numbers. As you know, the set of integers is adequate for many purposes, such as reporting the population of a country, the number of dollars you have (or owe), the number of vertices in a triangle, and so on. The integers alone are not suitable for other purposes, particularly for use in the process of measurement. If we had only the integers to use for measuring we would have to invent names for subdivisions of units. We do this to some extent; we speak either of 5 feet 4 inches, or $5\frac{1}{3}$ feet. But we do not use a different name for a subdivision of an inch. Instead, we use rational numbers, and speak of $7\frac{1}{4}$ inches, or 5.3 inches. If we had only the integers, we could never speak of $3\frac{1}{2}$ quarts, or 2.3 miles, or 0.001 inch.

Recall that a rational number may be defined as one which is the quotient of a whole number and a counting number. Thus, a rational number may be named by the fraction symbol $\frac{p}{q}$, where p and q are integers, and $q \neq 0$.

Just as there is a negative integer which corresponds to each positive integer (or counting number), there is a negative rational number which corresponds to each positive rational number.

You have already studied the properties of rational numbers, which may be summarized as follows:

Closure: If a and b are rational numbers, then $a + b$ is a rational number, $a \cdot b$ is a rational number, $a - b$ is a rational number, and $\frac{a}{b}$ is a rational number if $b \neq 0$.

Commutativity: If a and b are rational numbers, then $a + b = b + a$, and $a \cdot b = b \cdot a$.

Associativity: If a, b, and c are rational numbers, then $a + (b + c) = (a + b) + c$, and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Identities: If a is a rational number, then $a + 0 = a$, and $a \cdot 1 = a$.

Distributivity: If a, b, and c are rational numbers, then $a \cdot (b + c) = a \cdot b + a \cdot c$.

Additive inverse: If a is a rational number, then there is a number $(-a)$ such that $a + (-a) = 0$.

Multiplicative inverse: If a is a rational number and $a \neq 0$, then there is a number b such that $a \cdot b = 1$.

Order: If a and b are different rational numbers, then either $a > b$, or $a < b$.

Class Discussion Exercises 5-1

1. Answer questions (a) to (d), replacing "counting number" by "negative integer".

2. What set of integers has exactly the same properties as the set of counting numbers?

3. Express the following numbers in the form $\pm \frac{p}{q}$:

(a) $5\frac{3}{4}$ (b) 12 (c) $\left(8 - \frac{5}{6}\right)$ (d) $\left(-7 + \frac{1}{3}\right)$ (e) 72 (f) 0

(g) $3\frac{1}{2}$ (h) .47 (i) $\frac{-5}{-3}$ (j) $\left(6 + \frac{9}{10}\right)$

4. Recall that the same rational number may be named in various ways. Find a different name (which is also a fraction) for each of the rational numbers in problem 3.

5. Which of the rational numbers in Problem 3 are integers? positive integers?

6. How can you tell whether two fractions represent the same rational number? State two different ways.

7. Recall that the simplest name for a rational number $\frac{p}{q}$ is the one in which p and q have no common factor except 1. What is the simplest name for each of the rational numbers in problem 3?

8. If $\frac{p}{q}$ is the simplest name for a rational number which is also an integer, what must q be?

9. For each of the numbers in problem 3, tell of what two integers the rational number is the quotient. State your answer using the division sign " $\div$ ".

10. Is there more than one set of correct answers for problem 9?

11. Look at each statement below and tell which of the properties it illustrates.

a. $-\frac{2}{3} + \frac{5}{6} = \frac{1}{6}$, and $\frac{1}{6}$ is a rational number.

b. $\frac{5}{8} + 0 = \frac{5}{8}$.

c. $1 \cdot (-\frac{3}{4}) = -\frac{3}{4}$.

d. $(-\frac{3}{4}) \cdot (-\frac{5}{8}) = +\frac{15}{32}$, and $+\frac{15}{32}$ is a rational number.

e. $\frac{2}{3} \cdot \left(\frac{1}{3} + \frac{1}{2}\right) = \frac{2}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2}$

f. $\frac{5}{6} + -\frac{5}{6} = 0$

g. $(-\frac{5}{8}) \cdot (-\frac{3}{4}) = (-\frac{3}{4}) \cdot (\frac{5}{8})$

h. $\frac{7}{6} + (-\frac{2}{3}) = (-\frac{2}{3}) + \frac{7}{6}$

i. $\frac{11}{10} + \left(\frac{3}{10} + \frac{7}{10}\right) = \left(\frac{11}{10} + \frac{3}{10}\right) + \frac{7}{10}$

j. $\frac{2}{3} \cdot \left(\frac{1}{2} \cdot \frac{1}{5}\right) = \left(\frac{2}{3} \cdot \frac{1}{2}\right) \cdot \frac{1}{5}$

12. Which of the properties stated for the rational numbers does not hold for the set of integers? Illustrate each property which does hold.

13. Which of the properties do not hold for the positive integers? Illustrate each property which holds.

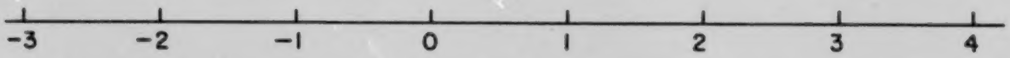
14. Give an illustration for each of the properties stated for the rational numbers. Use some positive and some negative numbers in your illustrations.

15. What is the additive inverse of $-\frac{7}{4}$? What is its multiplicative inverse? What is another name for "multiplicative inverse"?

5-2. Density of Rational Numbers

One of the observations you have made about the integers is that every integer is preceded by a particular integer, and is followed by a particular integer. The integer which precedes -8 is -9, and the integer which follows 1005 is 1006. In other words, if n is an integer, then its predecessor is $(n - 1)$, and its successor is $(n + 1)$.

This means that on the number line there are wide gaps between the points which correspond to the integers.



Now consider the rational numbers, and the points on the number line to which they correspond. Such points are called rational points. On the number line below are shown the rational points which may be represented by the fractions with denominators 2, 3, 4, and 6.

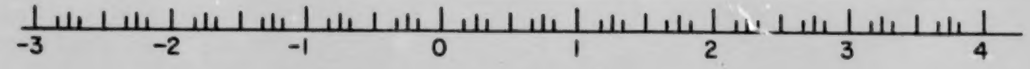


Figure 5-2

Exercises 5-2a

1. In figure 5-2, find the points which correspond to these numbers: $-\frac{6}{2}, -\frac{5}{2}, -\frac{4}{2}, -\frac{3}{2}, -\frac{2}{2}, -\frac{1}{2}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots$

2. Find the points that correspond to these numbers:

$$-\frac{7}{6}, -\frac{6}{6}, -\frac{5}{6}, -\frac{4}{6}, -\frac{3}{6}, -\frac{2}{6}, -\frac{1}{6}, \frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6} \ldots$$

3. Were any of the points in problems 1 and 2 the same point? If so, which ones?

4. Suppose there are already located points for the rational numbers represented by fractions with denominators 2, 3, 4, 5, and 6. You then locate points represented by fractions with denominator 7. How many new points (not already located) for sevenths will there be between the points for the integers 1 and 2? Between the points for 3 and 4?

5. Suppose that then you locate points for fractions with denominator 8. How many new points will there be between the points for any two consecutive integers?

6. Consider all rational points between 0 and 1 which are named by fractions with denominators 1 to 8 inclusive. These points are named below. The first row shows the fractions with denominator 1, the second row the fractions with denominator 2, the third row fractions (for new points) with denominator 3, and so on.

$$\frac{0}{1} \qquad \frac{1}{1}$$

$$\frac{1}{2}$$

$$\frac{1}{3} \qquad \frac{2}{3}$$

$$\frac{1}{4} \qquad \frac{3}{4}$$

$$\frac{1}{5} \qquad \frac{2}{5} \qquad \frac{3}{5} \qquad \frac{4}{5}$$

$$\frac{1}{6} \qquad \frac{5}{6}$$

$$\frac{1}{7} \qquad \frac{2}{7} \qquad \frac{3}{7} \qquad \frac{4}{7} \qquad \frac{5}{7} \qquad \frac{6}{7}$$

$$\frac{1}{8} \qquad \frac{3}{8} \qquad \frac{5}{8} \qquad \frac{7}{8}$$

(a) Why is $\frac{0}{3}$ omitted from the row for thirds?

(b) Why is $\frac{2}{4}$ omitted from the row for fourths?

(c) Why are there more new points named in the row for fifths and in the row for sevenths than in the row for sixths?

7. The rational numbers named in problem 6 are combined in one row below, and listed in order from smallest to largest.

$$\frac{0}{1}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{3}{5}\ \frac{5}{8}\ \frac{2}{3}\ \frac{5}{7}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{7}{8}\ \frac{1}{1}$$

Explain why the first six fractions should be in the order shown; the last six fractions.

8. The row of fractions (and the set of rational points the fractions name) could be increased by inserting next the fractions with denominator 9, then the fractions with denominator 10, and so on. How many new points would be named by fractions with denominator 9? With denominator 10? With denominator 11?

9. What kind of number as denominator seems to name the largest number of points not already named? Why?

We may follow a different method for naming and locating new points. Consider points for the fractions $\frac{1}{8}$ and $\frac{1}{7}$. We assume that a point halfway between these points should correspond to the rational number halfway between $\frac{1}{8}$ and $\frac{1}{7}$, - that is, to their average. Is this assumption reasonable? Verify the following computation:

$$\frac{1}{8} + \frac{1}{7}$$

$\frac{7}{56} + \frac{8}{56} = \frac{15}{56}$ $\qquad \frac{1}{2} \times \frac{15}{56} = \frac{15}{112}$. The average of $\frac{1}{8}$ and $\frac{1}{7}$ is $\frac{15}{112}$.

Are the differences of $\frac{15}{112}$ from $\frac{1}{7}$ and $\frac{1}{8}$ the same? That is, is the statement $\frac{1}{7} - \frac{15}{112} = \frac{15}{112} - \frac{1}{8}$ a true statement?

$$\frac{1}{7} - \frac{15}{112} = \qquad \frac{15}{112} - \frac{1}{8}$$

$$\frac{16}{112} - \frac{15}{112} = \frac{1}{112} \qquad \frac{15}{112} - \frac{14}{112} = \frac{1}{112}$$

Thus the point corresponding to $\frac{15}{112}$ would be the same distances from the points for $\frac{1}{7}$ and $\frac{1}{8}$. In other words, it would be the mid-point of the segment with end-points $\frac{1}{8}$ and $\frac{1}{7}$.

By finding mid-points in this manner it is possible to name a point midway between each pair of consecutive points named in the row of fractions in problem 7 of Exercises 5-2a.

10. The names for some new points found in this way are shown below, written below and between the fractions used for finding them. Find names for some more new points in the manner described. What fraction names the point midway between $\frac{1}{7}$ and $\frac{1}{6}$? Between $\frac{2}{5}$ and $\frac{3}{7}$? Between $\frac{6}{7}$ and $\frac{7}{8}$?

$$\frac{0}{1}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{3}{5}\ \frac{5}{8}\ \frac{2}{3}\ \frac{5}{7}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{7}{8}\ \frac{1}{1}$$

$$\frac{1}{16}\ \frac{15}{112} \qquad \frac{17}{48} \qquad \frac{49}{80} \qquad \frac{41}{56} \qquad \frac{71}{84}$$

11. If you found all the new points in this row that could be found in this way, how many would you find? You would then have a new row of fractions, which would begin as follows:

$\frac{0}{1}\ \frac{1}{16}\ \frac{1}{8}\ \frac{15}{112}\ \frac{1}{7}\ \frac{13}{42}$, etc. This process could be continued indefinitely. You could find points between $\frac{0}{1}$ and $\frac{1}{16}$, between $\frac{1}{16}$ and $\frac{1}{8}$, between $\frac{1}{8}$ and $\frac{15}{112}$, and so on.

The discussion above suggests an important property of the rational points on the number line. This is the property of density: Between any two distinct rationals there is a third rational. This means that the number of rational points on any segment is unlimited; no matter how many points on a very small segment have been named, it is possible to name as many more as you please.

Exercises 5-2b

1. Are the integers dense? That is, is there always a third integer between any two integers? Illustrate your answer.

2. Is there a smallest positive integer? a largest?

3. Is there a smallest negative integer? a largest?

4. Is there a smallest positive rational number? a largest negative rational number?

5. Think of the points for 0 and $\frac{1}{100}$ on the number line. Name the rational point which is halfway between 0 and $\frac{1}{100}$. Name the point halfway between the point you named and 0.

6. Could you continue to find other rational points in the manner described in problem 5?

7. Think of the segment with end-points $\frac{1}{1000}$ and $\frac{2}{1000}$. Show a plan you could follow to name as many points as you please on this segment. Use your plan to name at least five points.

5-3. Irrational Numbers

You know that in mathematics a line is thought of as a set of points. On the number line, many evenly spaced points correspond to the positive and negative integers. These points determine a set of line segments whose end-points correspond to the integers. On any one of these line segments, such as the segment with end-points 3 and 4, it is possible to name as many rational points as you please.

Even though the rational points are dense, are there still other points on the number line which can not be named by rational numbers?

Here is a way to find such a point geometrically.

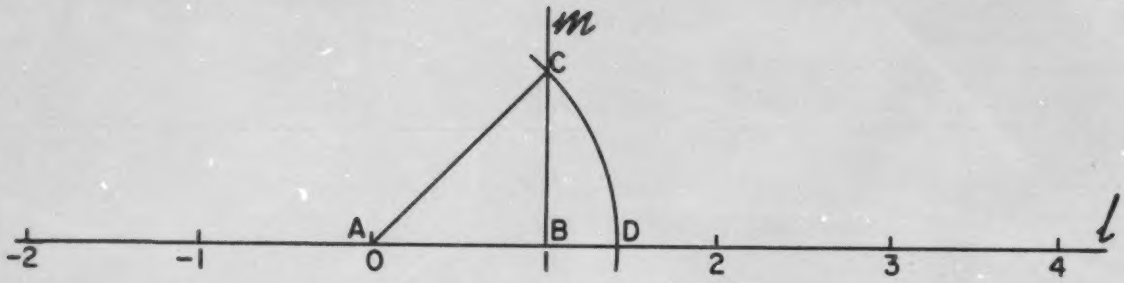
a. Draw a line ℓ and on it lay out a number line. Call the point 0 point A, and the point 1 point B.

b. At B, construct a ray m perpendicular to ℓ.

c. On m lay off a line segment $\overline{BC}$, one unit long.

d. Draw segment $\overline{AC}$.

e. With A as center and radius AC, draw a circular arc which intersects ℓ. Call the point of intersection point D.



Now consider two questions:

(1) To what number (if any) does point D correspond?

(2) Is this number a rational number?

Consider the first question, "To what number does point D correspond?" First find the length of $\overline{AC}$, since $\overline{AC}$ and $\overline{AD}$ have the same length. We shall use as unit of measure the unit distance on the number line. In Figure 5-3, what kind of triangle is triangle ABC? What is the measure of $\overline{AB}$? of $\overline{BC}$? What property can be used to find the measure of $\overline{AC}$? Explain the following statements:

$$AC^2 = BC^2 + AB^2$$

$$AC^2 = 1^2 + 1^2$$

$$AC^2 = 2$$

$$AC = \sqrt{2}, \text{ so}$$

$AD = \sqrt{2}$. Therefore, the point D corresponds to the number $\sqrt{2}$. Is $\sqrt{2}$ a rational number? Is it the quotient of two integers, and can it be represented as a fraction $\frac{p}{q}$, in which p and q are integers and $q \neq 0$?

To answer this question, we shall follow a line of reasoning which mathematicians (and other people) use very often. We shall assume that $\sqrt{2}$ is a rational number, and then show that this leads us to a conclusion which could not possibly be true; and that, therefore, the statement "$\sqrt{2}$ is a rational number" must certainly be false.

Before we begin we need to recall certain properties of numbers, and of squares and square roots of numbers.

1. If a is an integer, then a^2 is an integer.
2. If $a = b^2$, then b is the square root of a (written $\sqrt{a}$).
3. An even number is an integer which has the factor 2. Any even number may be written in the form $2 \cdot n$, where n is an integer.
4. An odd number is a number which is 1 greater than an even number. Any odd number may be written in the form $(2 \cdot n + 1)$, where n is an integer.
5. The square of an even number is an even number. The square of an odd number is an odd number.
6. If the square root of an even number is an integer, then it is an even integer. (If it were not even it would be odd, and then its square would be odd.)
7. The simplest name for a rational number is a fraction $\frac{p}{q}$ in which p and q are integers, $q \neq 0$, and p and q have no common factors except 1.

Find illustrations for these properties, so you are sure that you understand them.

$\sqrt{2}$ Is Not a Rational Number

Is $\sqrt{2}$ a rational number? Assume that $\sqrt{2}$ is a rational number.

Then $\sqrt{2} = \frac{p}{q}$, where p and q are integers, $q \neq 0$, and take $\frac{p}{q}$ as the simplest name for the number.

If $\sqrt{2} = \frac{p}{q}$, then $2 = \frac{p^2}{q^2}$

and $p^2 = 2\,q^2$. Since p and q are integers, then p^2 and q^2 are integers.

Since $p^2 = 2\ q^2$, then p^2 is an even number.

So p is also an even number, and may be written as $2\cdot a$, where a is an integer.

Then $(2\cdot a) = 2\cdot q^2$

and $2\cdot a\cdot 2\cdot a = 2\cdot q^2$

and $2\cdot(2\cdot a^2) = 2\cdot q^2$

and $2\cdot a^2 = q^2$. This tells us that q^2 is an even number, so q is also an even number. And we have already shown that p is an even number.

Thus our assumption that $\sqrt{2}$ is a rational number with $\frac{p}{q}$ its simplest name has led us to the conclusion that p and q both have the factor 2. This is impossible, since the simplest name for a rational number is the one in which p and q have no common factor other than 1. So the statement " $\sqrt{2}$ is a rational number" must be false.

Explain which property justifies each statement in this proof.

Since the measure of segment AD in Figure 5-3 is $\sqrt{2}$, then $\sqrt{2}$ must be the number which corresponds to point D. It has been shown that $\sqrt{2}$ is not a rational number. Therefore, there is at least this one point on the number line which corresponds to some number which is not a rational number. In other words, even though the rational points are dense, the set of points on the number line contains more points than there are rational numbers.

A number like $\sqrt{2}$, which is not a rational number, is called an irrational number. Do you see the word "ratio" in the word "rational"? The prefix "ir" changes the meaning of "rational" to "not rational."

Exercises 5-3

1. Construct a figure like figure 5-3, and label point D "$\sqrt{2}$". Then use your compass to locate the point which corresponds to the number ($-\sqrt{2}$), and label it.

2. Lay out a number line, using a unit of the same length as the unit in problem 1. Call the point 0 point A, and the point 2 point E. At E construct a line segment perpendicular to the number line and 1 unit in length, and call it $\overline{EF}$. Draw $\overline{AF}$. What is the measure of segment $\overline{AF}$?

3. Use the drawing for problem 2, and locate on the number line the points which correspond to $\sqrt{5}$ and $-\sqrt{5}$. Label the points.

4. Do you think $\sqrt{5}$ is a rational number or an irrational number? Why?

*5. Using the same method as in problems 2 and 3, locate the point $\sqrt{3}$. Can you work out a way to locate the point for $\sqrt{6}$? $\sqrt{7}$?

6. Locate the points which correspond to these numbers:
$(2\cdot\sqrt{2})$; $(3\cdot\sqrt{2})$; $(-3\cdot\sqrt{2})$.

7. Do you think that $(2\sqrt{2})$ is a rational number or an irrational number?

*8. BRAINBUSTER. Prove that $\sqrt{5}$ is an irrational number.

Enumerating the Rationals.

In the preceding section it was proved that $\sqrt{2}$ is not a rational number. Moreover, it appears that there are many other numbers, such as $\sqrt{3}$, $\sqrt{5}$, and also π which are not rationals. If you think about the rationals and the irrationals a bit you see that there are many "more" irrationals than rationals. For example, every number of the form $\frac{a}{b}\sqrt{2}$, when $\frac{a}{b}$ is rational, will be an irrational. Hence the set $\left\{\frac{a}{b}\sqrt{2}\right\}$ can be put into 1-1 correspondence with the set of rationals $\left\{\frac{a}{b}\right\}$. Yet the set $\left\{\frac{a}{b}\sqrt{2}\right\}$ is obviously only a very small part of the irrationals!

Indeed, we have suffered a great disillusionment - the rational numbers, despite being dense on the number line, actually leave empty more positions than the fill! An even worse shock to our intuition, perhaps, is to find that we can so easily construct a line segment whose length is not given by a rational number.

In fact, one of the really important distinctions between the rational number system and the system of irrationals is that you can show how to display or enumerate all the rationals. One scheme is to proceed as follows. Write the array

$$\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & . & . & . \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \frac{7}{2} & . & . & . \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \frac{7}{3} & . & . & . \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \frac{7}{4} & . & . & . \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \frac{7}{5} & . & . & . \\
\frac{1}{6} & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & .
\end{array}$$

Clearly we can find the positive rational $\frac{4}{5}$ in the 4th column at the fifth row. In what row and what column would you look for the rational $\frac{9}{10}$? For $\frac{8}{17}$? For $\frac{p}{q}$?

In order to enumerate the positive rationals in a systematic fashion we could proceed as indicated by the snaky line, starting with 1 and proceeding successively down through the array. The negative rationals can be described by a similar array, and zero can easily be added.

5-4. A Decimal Representation for $\sqrt{2}$

Numbers like π and $\sqrt{2}$ correspond to points on the number line, they specify lengths of line segments and they satisfy our natural notion of what a number is. Perhaps the most unusual aspect about $\sqrt{2}$ is the way it was defined: $n = \sqrt{2}$ is the positive number which when squared yields 2, so that

$$n^2 = 2.$$

This differs from our previous way of defining numbers, since up to now we have dealt mainly with integers and numbers defined as ratios of integers.

In order to help us gain a better understanding of $\sqrt{2}$ we shall look for a new way of describing $\sqrt{2}$ in terms of more familiar notions. If, for example, we could somehow express $\sqrt{2}$ as a decimal this would help us to compare it to the rational numbers we know. It would also tell us where to place it on the number line.

Let us think about the definition of the number $n = \sqrt{2}$, namely $n^2 = 2$ or $(\sqrt{2})^2 = 2$. If we think of squaring 1 and 2 we note immediately that

$$1^2 < (\sqrt{2})^2 < 2^2 \text{ and hence } 1 < \sqrt{2} < 2$$

This tells us that $\sqrt{2}$ is greater than 1 and less than 2, but we already knew that. We might try a closer approximation by testing the squares of 1.1, 1.2, 1.3, 1.4, 1.5. A little arithmetic of of this sort (try it, it is good for your soul!) leads us to the result

$1.96 = (1.4)^2 < (\sqrt{2})^2 < (1.5)^2 = 2.25$, and therefore we conclude that $1.4 < \sqrt{2} < 1.5$.

The arithmetic involves a little more work at the next stage but we see with a little more computation that

$$1.9881 = (1.41)^2 < (\sqrt{2})^2 < (1.42)^2 = 2.0164,$$

and therefore

$$1.41 < \sqrt{2} < 1.42.$$

If we try to extend the process further we get at the next stage

$$1.414 < \sqrt{2} < 1.415.$$

You can see that this process can be continued as long as our enthusiasm lasts, and gives a better decimal approximation at every stage. If we continued to 7 place decimals we would find

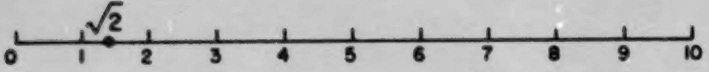
$1.4142135 < \sqrt{2} < 1.4142136$. This is a very good approximation to $\sqrt{2}$, for $(1.4142136)^2 = 2.00000010642496$.

By the use of the defining property, $n^2 = 2$, then, we can find decimal approximations for $\sqrt{2}$ which are as accurate as we wish. We are led to write

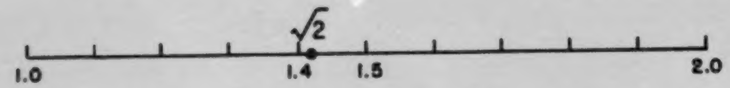
$$\sqrt{2} = 1.4142135 \ldots$$

where the three dots indicate that the digits continue without terminating, as the process above suggests.

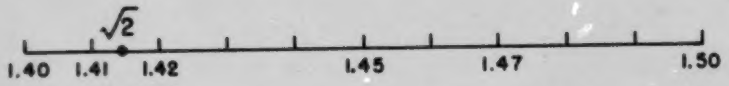
Geometrically the procedure we have followed can be described as follows in the number line. Looking first at the integers of the number line on the segment from 0 to 10, we saw that $\sqrt{2}$ would be between 1 and 2.



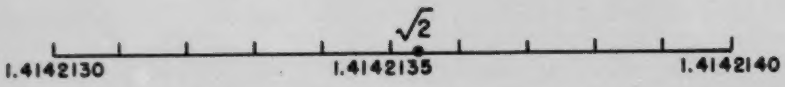
Enlarging our view of this interior (by a ten-fold magnification) we saw that $\sqrt{2}$ is on the segment with end-points 1.4 and 1.5



and again magnifying this picture, $\sqrt{2}$ lies within the interior (1.41, 1.42)



and so on till the 8th stage shows us that $\sqrt{2}$ lies between 1.4142135 and 1.4142136,



This process shows us how to read off the successive digits in the decimal representation for $\sqrt{2}$. At the same time it gives a way to define the position of the point in the real line.

When we write the number $\sqrt{2}$ as 1.4142135 ... it looks suspiciously like many rational numbers we have seen, such as

$\frac{1}{3} = .3333333 \ldots \qquad \frac{1}{7} = .14285714 \ldots$

We pause to ask, how are they different and how can we tell a rational from an irrational number when we see only the decimal representations of the numbers?

Exercises 5-4

Between what two integers are the following irrational numbers? (Write your answer as suggested for problem 1.)

1. $\sqrt{30}$ [$\underline{\ \ ?\ \ } < \sqrt{30} < \underline{\ \ ?\ \ }$]
2. $\sqrt{89}$
3. $\sqrt{253}$

4. $\sqrt{4280}$ (Hint: 4280 is $42.80 \cdot 10^2$, so begin estimating by thinking of $\sqrt{43} \cdot 10$)

5. $\sqrt{9315}$

6. Find simple names for

   (a) $(\sqrt{3})^2$

   (b) $(1.732)^2$

   (c) $(1.733)^2$

   (d) Find the difference between your answers for parts (a) and (b); find the difference between your answers for parts (a) and (c).

   (e) To the nearest thousandth what is the best decimal expression for $\sqrt{3}$?

Which of the decimals suggested is the better approximation of the following irrational numbers?

7. $\sqrt{3}$: 1.73 or 1.74

8. $\sqrt{15}$: 3.87 or 3.88

9. $\sqrt{637}$: 25.2 or 25.3

Find, to the nearest tenth, the nearest decimal expression for these irrational numbers:

10. $\sqrt{10}$ 11. $\sqrt{149}$ 12. $\sqrt{221}$

*13. For what number n^2 is $n = 10$?

*14. For what number n^2 is $n = 149$?

5-5. Decimal Representations for the Rational Numbers

Some rational numbers are almost immediately convertible to decimal form. We know how to write, by inspection,

$\frac{1}{2} = .5$, $\frac{1}{4} = .25$, $\frac{1}{8} = .125$, $\frac{1}{5} = .2$, $\frac{1}{25} = .04$

$\frac{1}{125} = .008$, and also $\frac{17}{2} = 8.5$, $\frac{53}{25} = 2.12$, $\frac{75}{8} = 9.375$. Rationals of this form can easily be expressed as terminating decimals.

For other rational numbers, a decimal expression may not be so obvious but we can always obtain it by the usual process of division. For example

$$\frac{1}{3} = .33333\ \ldots,\quad \frac{8}{3} = 2.6666666\ \ldots$$

$$\frac{1}{7} = .142857142857142857\ \ldots$$

$$\frac{1}{11} = .09090909$$

$$\frac{1}{13} = .076923076923076923 07\ \ldots$$

$$\frac{123}{14} = 8.7857142857142\ \ldots$$

The examples we have discussed seem to suggest that the decimal expansions for rationals are either terminating (like $\frac{1}{2} = .5$) or non-terminating but repeating (like $\frac{1}{3} = .3333333\ \ldots$). What would be a reasonable way to investigate such a question? Since we have used the division of numerator by denominator to obtain a decimal representation we might study carefully the process which we carry out in such cases.

Consider the rational number 7/8. If we carry out the indicated division in decimal form we write

$$\begin{array}{r l}
.875 & \\
8\overline{)\,7.000} & \\
\underline{6\,4} & \\
60 & \text{remainder } 6 \\
\underline{56} & \\
40 & \text{remainder } 4 \\
\underline{40} & \\
0 & \text{remainder } 0
\end{array}$$

In dividing by 8, the only remainders which can occur are 0, 1, 2, 3, 4, 5, 6, and 7. The only remainders which did occur were 6 at the first stage, then 4 and finally 0. When the remainder 0 occurs, the division is exact, the process terminates and the rational number has a terminating decimal expansion. You can easily check this for the examples of terminating decimals given at the beginning of this section.

But what about a rational number which does not have a terminating decimal representation? Suppose we look at a particular example of this kind, say 2/13. The process of dividing 2 by 13 proceeds like this:

```
        .153846153
   13 ) 2.000000000
        1 3                remainder
          70                   7
          65
           50                  5
           39
           110                11
           104
             60                6
             52
              80               8
              78
               20              2
               13
                70             7
                65
                 50            5
                 39
                 110          11
                 etc.
```

Here the possible remainders are 0, 1, 2, 3, 4, ..., 10, 11, 12. Not all the remainders do appear, but 7, 5, 11, 6, 8, 2 occur first in this order. At the next stage in the division the remainder 7 reoccurs and the sequence of remainders 7, 5, 11, 6, 8, 2 is then certain to occur again. In fact the process repeats itself again and again without stopping. The corresponding sequence of digits in the quotient - 153846 - will therefore occur periodically in the non-terminating decimal expansion for 2/13.

In order to write such a periodic decimal concisely and without ambiguity it is customary to write

$.1538461538461538461 \ldots = .\,\overline{153846} \ldots$

where the bar (vinculum) over the digit sequence 153846 indicates that this set of digits repeats indefinitely in the expansion.

The method we have discussed is quite a general one and it can be applied to any rational number a/b. If the indicated division is performed then the only possible remainders which can occur are 0, 1, 2, 3, ... (b - 1). We look only at the stages which contribute to the digits following the decimal point in the quotient. These stages occur after we begin to add zeros to the dividend. If the remainder 0 occurs, the decimal expansion terminates at this stage in the division process and the rational number has a terminating decimal representation. (Note that a zero remainder may occur prior to this stage without terminating the process, for example, in 511/5 = 102.2.) If 0 does not occur as a remainder after the decimal point stage, then after at most (b-1) steps in the division process one of the possible remainers 1, 2, ... (b - 1) will reoccur and the digit sequence will start repeating.

We see from this argument that any rational number has a decimal expansion which is either terminating or periodic and non-terminating.

Actually, we may write a terminating decimal expansion like .25 as .25000 ... or $.25\overline{0}$ ..., with a repeated zero to provide a periodic expansion. In fact we can also write .25 as periodic decimal in a second way, namely

$$.25 = .2499999\ \ldots = .24\bar{9}\ \ldots$$

After making this simple agreement, which allows us to consider a terminating decimal as periodic, we can say quite concisely:

Any rational number has a periodic non-terminating decimal representation.

Exercise 5-5

Find decimals for these rational numbers. Continue the division until repeating begins, and write your answer to at least ten decimal places.

1. 14/37 2. 3/7 3. 9/4 4. 1/41 5. 11/909
6. 5/54 7. 128/125 8. 3/35 9. 1/82 *10. 1/17

11. (a) Which of the decimals in problems 1-9 terminates?
    (b) Write in completely factored form the denominators of the two rational numbers in problems 1-9 whose decimals terminate. What do you observe?

12. (a) Compare your decimals for problems (2) and (8). What do you observe? Are the denominators related in any way?
    (b) Compare your decimals for problems (4) and (9). Are the denominators related in any way?

5-6. The Rational Number Corresponding to a Periodic Decimal

We saw how easy it is to find by division the decimal expansion of a given rational number. But suppose we have the opposite situation, that is, we are given a periodic decimal. Does such a decimal in fact represent a rational number? How can we find out?

We can see how to approach this problem by considering a simple example. Let us write a periodic decimal, say .132132132132 ...

and call it n, so that $n = .132132132132 \ldots$ The periodic block of digits is 132. If we multiply n by 1000 this shifts the first block to the left of the decimal point and gives the relation

$$1000n = 132.132132132 \ldots$$

Since $n = .132132132 \ldots$

we can subtract n from each side of the first equation to yield

$$999n = 132 \quad \text{so that}$$

$$n = \frac{132}{999}, \quad \text{or in simplest terms,}$$

$$n = \frac{44}{333}.$$

We find by this process that $.132132132132 \ldots = \frac{44}{333}$.

The example we have carried out here illustrates a general procedure which mathematicians have developed to show that every periodic non-terminating decimal represents a rational number.

We see, therefore, that there is a one-to-one correspondence between the set of rational numbers $\left\{\frac{a}{b}\right\}$, $b \neq 0$ and the set of periodic decimal expansions $\{\pm a_1 a_2 a_3 \ldots a_k \cdot b_1 b_2 \ldots b_m \overline{c_1 c_2 \ldots c_n}\}$ where $a_1, a_2, \ldots, b_1, b_2, \ldots c_1, c_2, \ldots$ are digits 0, 1, 2, ..., 7, 8, 9.

It would be quite equivalent then for us to define the rational numbers as the set of all such periodic decimals.

The fact that a decimal representation for a rational must be periodic tells us something further about the decimal representation we found earlier for the irrational number $\sqrt{2}$. We can now be sure that the decimal representation for $\sqrt{2}$ must be a non-terminating, non-periodic decimal.

Why? For if it were periodic or terminating it would be a rational and this we know to be untrue.

What rationals have terminating decimal expansions? Before we leave the subject of decimals we want to discuss one interesting fact about terminating decimals.

We saw that rationals like $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, $\frac{15}{8} = 1.875$ $\frac{397}{1000} = .397$, $\frac{692}{25} = 27.68$ all are represented by terminating decimals. How can we determine when this will be the case? If we look for inspiration at the rationals of this type which we discussed we see an obvious clue: the denominator seem to contain only powers of 10 or powers of 2 or powers of 5.

Consider any rational number which has a power of 2 in the denominator, like

$\frac{39}{16} = \frac{39}{2^4}$. By multiplying by $1 = \frac{5^4}{5^4}$ we can write

$$\frac{39}{2^4} = \frac{(39)5^4}{2^4 \cdot 5^4} = \frac{39 \cdot 5^4}{10^4} = \frac{39(625)}{10000} = \frac{24375}{10000} = 2.4375$$

Similarly if we have a rational with a power of 5 in the denominator we can proceed as in the following example

$$\frac{3}{3125} = \frac{3}{5^5} = \frac{3 \cdot 2^5}{5^5 \cdot 2^5} = \frac{3 \cdot 32}{10^5} = \frac{96}{100000} = .00096.$$

Quite generally, if we have any rational number with only powers of 2 and powers of 5 in the denominator, we can use the same technique. For example,

$$\frac{3791}{2^7 \cdot 5^4} = \frac{(3791) \cdot 5^7 \cdot 2^4}{(2^7 \cdot 5^7)\ (5^4 \cdot 2^4)} = \frac{(3791) \cdot 5^7 \cdot 2^4}{10^7 \cdot 10^4} = \frac{(3791) \cdot 5^7 \cdot 2^4}{10^{7+4}} = \frac{(3791) \cdot 5^7 \cdot 2^4}{10^{11}}$$

= a terminating decimal.

In order to establish a general fact of this kind suppose we ask the following question. When is a rational number p/q (p and q assumed to have no common factor) representable as $\frac{N}{10^k}$ where N is an integer?

Suppose that this is indeed the case and that

$$\frac{p}{q} = \frac{N}{10^k} .$$

Therefore $q \cdot N = p \cdot 10^k$.

This says that q divides the product of p and 10^k. But we assumed that p and q have no factor in common. Hence q must divide 10^k. But the only possible factors of $10^k = 2^k 5^k$ are powers of 2 or powers of 5 or powers of 10.

Thus, we have proved that a rational number r has a terminating decimal representation if and only if the denominator of r consists only of powers of 2 and powers of 5; that is, r must be of the form

$$r = \frac{A}{2^m 5^n}$$

Exercises 5-6

What rational numbers have these decimal expressions:

1. $.\overline{09}\ldots$
2. $.\overline{1}\ldots$
3. $.05\overline{5}\ldots$
4. $.03\overline{3}\ldots.$
5. $.\overline{123}\ldots$
6. $.1625$
7. $.16\overline{6}\ldots$
8. Write the denominator of these rational numbers in completely factored form.

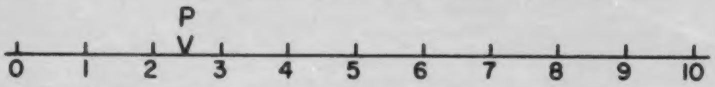
   (a) 7/32 47/100 15/14 3/25 17/50 5/9 11/6

   (b) Which of the numbers in part (a) have decimals which terminate?

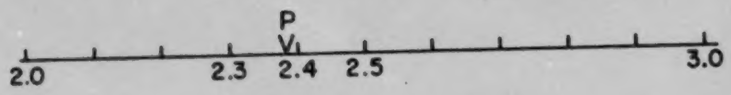
5-7 Rational Points on the Number Line

If we think of the rational numbers as specified by decimal representations, we see immediately how to locate and how to order the corresponding points on the number line.

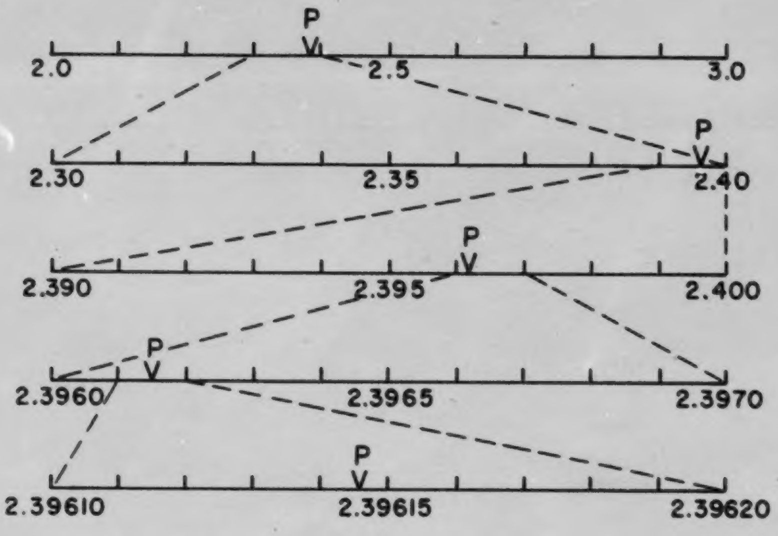
Consider for example the rational number $2.39\overline{614}\ldots.$ The unit digit 2 tells us immediately that the corresponding rational point P lies between the integers 2 and 3 on the number line. Graphically then the first rough picture is this.



A more precise description is obtained by looking at the first two digits 2.3 which tell us immediately that P lies between 2.3 and 2.4. On the unit interval from 2 to 3, then, divided into tenths (and magnified ten times for easy comparison) we find P as listed below.


P
2.0
2.3
2.4
2.5
3.0

If we continue the process of successively refining the location of P on the number line we have a picture such as the following:


P
2.0
2.5
3.0
P
2.30
2.35
2.40
P
2.390
2.395
2.400
P
2.3960
2.3965
2.3970
P
2.39610
2.39615
2.39620

From such a decimal representation for a rational then we easily find how to locate the number to any desired degree of accuracy on the number line.

Moreover, given any two distinct rationals in this form it is a simple matter to tell by inspection which is larger and which is smaller, and which precedes the other on the number line.

If you think of locating the point $\frac{1}{3}$ carefully on the number line would you prefer to use $\frac{1}{3}$ or .3333 ...? If you wish to compare $\frac{1}{3}$ with another rational, which form is easier to use, - or .33333...?

Exercises 5-7

1. Arrange each group of decimals in the order in which the points to which they correspond would occur on the number line. List first the point farthest to the left.

   (a) 1.379 1.493 1.385 5.468 1.372

   (b) -9.426 -2.765 -2.761 -5.630 -2.763

   (c) .15475 .15467 .15463 -.15475 .15598

2. In problem 1c, which points lie on the following segments:

   (a) The segment with endpoints 1 and 2?

   (b) The segment with endpoints 0 and 1?

   (c) The segment with endpoints .1 and .2?

   (d) The segment with endpoints .15 and .16?

   (e) The segment with endpoints .154 and .155?

3. Suppose you have a number line in which the distance from the point 0 to the point 1 is 10 centimeters long.

   Draw a 10 centimeter segment; label the endpoints 0 and 1, and divide the segment into tenths. Mark and label the following points:

   (a) .23 (b) .49 (c) .80 (d) .6 (e) .08 (f) .95

5-8. Definition of Irrational Numbers and Definition of Real Numbers

We have seen that all rational numbers have periodic decimal representations. We saw also that $\sqrt{2}$ is not rational and that it is represented by a non-periodic, non-terminating decimal. We called $\sqrt{2}$ an irrational number.

We now use this decimal form to define the set of irrational numbers. We define an irrational number as any number with a non-terminating, non-periodic decimal representation of the form

$$\pm\ a_1\, a_2 \dots a_k .\ b_1\, b_2\, b_3 \dots$$

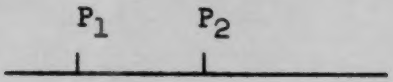
The system composed of all rational and irrational numbers we call the *real number system.*

From this we see that *any real number can be characterized by a decimal representation of the form*

$$\pm\ a_1\, a_2\, a_3 \dots a_k .\ b_1\, b_2 \dots$$

If the decimal representation is periodic it defines a rational number, otherwise the number is an irrational number.

With every point P on the real number line we associate one and only one number of this form by a process of successive location in decimal intervals of decreasing length. Note that any two distinct points P and P will correspond to distinct decimal representations, for if they occur as



on the number line we need only subdivide the number line by a sufficiently fine decimal subdivision to assure that P_1 and P_2 lie in segments defined by different decimal intervals.

Conversely, given any decimal we have found how to locate the corresponding point of the real number line by considering successive rational decimal approximations provided by the number.

Thus the real numbers possess a one-to-one correspondence with the points of the real number line.

5-9 Properties of the Real Number System

The real number system which we have now defined possesses all of the properties of the rationals and an additional very important one. We list first the properties which are also possessed by the system of rationals.

Property 1. Closure.

a) Closure under Addition (+). The real number system is closed under the operation of addition, i.e., if a and b are real numbers then $a + b$ is a real number.

b) Closure under Subtraction (-). The real number system is closed under the operation of subtraction, the inverse of addition, i.e., if a and b are real numbers then $a-b$ is a real number.

c) Closure under Multiplication ($\cdot$). The real number system is closed under the operation of multiplication, i.e., if a and b are real numbers then $a \cdot b$ is a real number.

d) Closure under Division ($\div$). The real number system is closed under the operation of division, the inverse of multiplication, i. e., if a and b are real numbers then $a \div b$ (when $b \neq 0$) is a real number.

The operations of addition, multiplication, subtraction, and division on real numbers display the properties which we have already observed for rationals. These may be summarized as follows:

Property 2. Commutativity.

a) If a and b are real numbers, then $a + b = b + a$.

b) If a and b are real numbers, then $a \cdot b = b \cdot a$.

Property 3. Associativity.

a) If a, b, and c are real numbers, then $a + (b+c) = (a+b) + c$.

b) If a, b, and c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Property 4. Identities.

a) If a is a real number, then $a + 0 = a$, i.e., zero is the identity element for the operation of addition.

b) If a is a real number, than $a \cdot 1 = a$, i.e., one is the identity element for the operation of multiplication.

Property 5. Distributivity. If a, b, and c are real numbers, then $a \cdot (b + c) = a \cdot b + a \cdot c$.

Property 6. Inverses.

a) If a is a real number, there is a real number (-a) such that $a + (-a) = 0$.

b) If a is a real number and $a \neq 0$ there is a real number b such that $a \cdot b = 1$.

Property 7. Order. The real number system is ordered, i.e., if a and b are different real numbers than either $a < b$ or $a > b$.

Property 8. Density. The real number system is dense, i.e., between any two distinct real numbers there is always another real number. Consequently, between any two real numbers we can find as many more real numbers as we wish. In fact we easily see that: 1) There is always a rational number between any two distinct real numbers, no matter how close. 2) There is always an irrational number between any two distinct real numbers, no matter how close.

The ninth property of the system of real numbers is one which is not shared by the rationals.

Property 9. Completeness. The real number line system is complete, i.e., to each point on the number line there corresponds a real number, and, conversely, to each real number there corresponds a point on the real number line.

We saw that in the system of rationals there is no number $n = \sqrt{2}$ which when squared yields 2. However, in the real number system as defined, such a number is included.

It happens that the nth root of any rational number which is not itself a perfect nth power is an irrational number. This means that such numbers as $\sqrt{3}$, $\sqrt[3]{\frac{5}{17}}$, $\sqrt[4]{15}$, $\sqrt{\frac{2}{3}}$ are irrational numbers. ($\sqrt[4]{15}$ means a number n such that $n^4 = 15$.) Hence, in the system of rationals we cannot hope to extract nth roots of any numbers which are not perfect nth powers. However, when we adjoin the irrationals to form the real number system we see that we have added all these missing nth roots of rational numbers. Thus a very useful property of the real number system is:

The real number system contains the nth roots, $\sqrt[n]{\frac{a}{b}}$ of all positive rational numbers a/b, $b \neq 0$.

This assures us that we can find among the real numbers such numbers as $\sqrt{3}$, $\sqrt{7}$, $1 + \sqrt{5}$, $\sqrt[3]{4}$, $\sqrt[5]{23}$, and any other such number or combination of roots.

In addition to irrational numbers which arise from extracting roots of rational numbers there are many more irrational numbers like π which are called transcendental irrational numbers. When you study logarithms in high school, you will be studying numbers which are almost all transcendental irrational numbers. If N is any positive real number and x is the index of the power to which 10 must be raised to yield N, or

$$10^x = N$$

then we say that x = the logarithm of N to the base 10. If N is the power of 10, say $N = 10^2$, then clearly $10^x = 10^2$ and $x = 2$ = the logarithm of 10^2 to the base 10. In such a case, the logarithm is a rational number. But for most numbers the logarithm will be a (transcendental) irrational number.

The trigonometric ratios, sine of an angle, and tangent of an angle are other expressions which usually turn out to be transcendental rational numbers. These ratios are defined in Unit 6.

Exercises 5-9

1. Which of the following numbers are rational and which are irrational? Make two lists.
   a. $.231\overline{231}\ldots$
   b. $.23123112311123\ldots$
   c. $\frac{3\sqrt{2}}{7\sqrt{2}}$
   d. $\sqrt{7}$
   e. $.78\overline{342}\ldots$
   f. $\frac{\pi}{2}$
   g. $\frac{3}{4}\sqrt{6}$
   h. $9 - \sqrt{3}$
   i. $.75000$
   j. $\frac{58}{11}$
2. Write each of the rational numbers in problem 1 as a decimal numeral and as a fraction.
3. For each of the irrational numbers in problem one write a decimal correct to the nearest hundredth.
4. a. Make up 3 terminating decimals for rational numbers.
   b. Make up 3 non-terminating decimals for rational numbers.
   c. Make up 3 decimals for irrational numbers.

You have learned how to insert other rational numbers between two given rationals. Now that you have studied decimal representations for real numbers, you can see how to insert rational or irrational numbers between real numbers. Look at these decimals for two numbers a and b.

$a = 4.219\overline{317}\ \ldots$

$b = 4.23401001000100001\ldots$

Are these decimals for rational or irrational numbers?

These numbers are quite close together, but any decimal which begins 4.22... will be greater than a and less than b. We can then continue the decimal in such a way as to make it rational or to make it irrational. For example, $4.225\overline{225}\ \ldots$ is rational and 4.225622566225666... is irrational.

5. a. Write a decimal for a rational number between $2.3846\overline{846}$ and $2.369\overline{369}$.

   b. Write a decimal for an irrational number between the decimals in part (a).

6. Write decimals for (a) a rational number and (b) an irrational number between .346019... and .342806...

7. Write decimals for (a) a rational number and (b) an irrational number between $67\overline{283}\ \ldots$ and 67.28106006...

5-10. Rational Approximations to Irrationals

Whenever we give an irrational number in its decimal form, for example, an irrational number beginning, N = .019234675... , we see that we automatically define a sequence of rational numbers which approximate closer and closer to the irrational number N. We can read off such a sequence of rational approximations as,

.01

.019

.0192

.01923

.019234

.0192346

.01923467

.019234675

Practically speaking, this is often how we compute, by using a rational approximation to the irrational number. On the other hand mathematicians and other scientists frequently need to use the exact irrational number such as

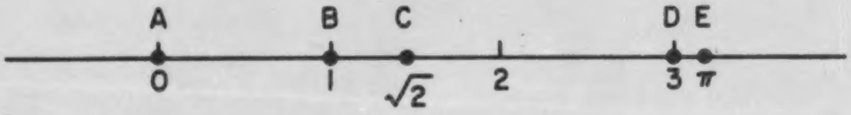
$$\sqrt{2}, \quad \sqrt{3} \quad \pi \quad \sqrt[3]{\pi} \quad \sqrt{3} \quad -\sqrt{2}$$

in order to get a final exact result.

Question: What property of the real number system is it which assures us that any real number can be approximated as closely as we please by a rational number?

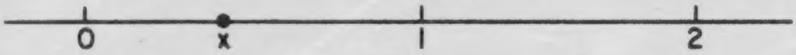
Geometric Properties of the Real Line

The 1-1 correspondence between the real numbers and the points of the real number line gives us for the first time a satisfactory geometric representation. We know now that there are no gaps or missing points in the real line. We can speak of tracing the real number line continuously and know that the segment described at any stage has a length which is measured by a real number. Thus in the number line indicated below we know that $\overline{BC}$ has a length



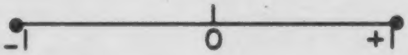
of measure $\sqrt{2} - 1$, the length of $\overline{CD}$ has measure $3 - \sqrt{2}$, $\overline{BE}$ has measure $\pi - 1$, $\overline{CE}$ is measured by $\pi - \sqrt{2}$.

We can think of a point moving continuously from 0 to 1 and associate with it in every position a



real number. Because of this continuous property of our real number system, we sometimes refer to it as the continuum of real numbers.

If we think of the geometric line segment from -1 to +1, including the endpoints, it can be described completely as the set of all real points x for which $-1 \leq x \leq 1$,



5-11. Rationals and Irrationals in the World Around Us

We see many examples of rationals every day--the price of groceries, the amount of a bank balance, the rate of pay, the amount of a weekly salary, the grade on a test paper.

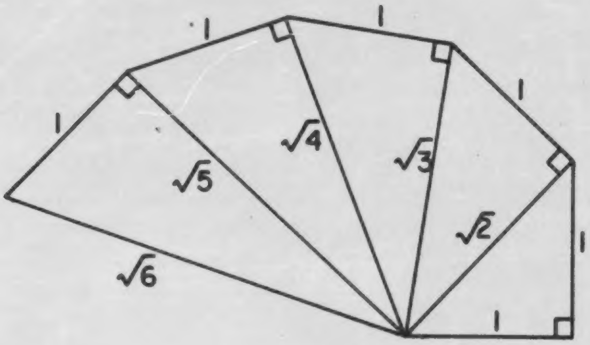
Although we have not considered the irrationals for very long, it is easy to see many examples which involve irrational numbers. For example, consider a circle of radius one unit. What is its length? Why 2π units of course. In fact, any circle whose radius is a rational number has a circumference which is irrational. Also, the interior of such a circle of radius r has an area which is measured by an irrational number (πr^2).

The volume V of a circular cylinder, you may recall, is given as $V = \pi r^2 h$ and its lateral surface area A is

$$A = 2 \pi rh$$

where h is the altitude of the cylinder. Here also the volume and area are given by irrational numbers if the radius r and altitude h are given by rationals.

Also, we note how easy it is to construct lengths of irrational measure by the following simple succession of right triangles:



What other examples of irrational numbers can you name?

UNIT 6

SIMILAR TRIANGLES

6-1. The Graph of $y = 2x$ (First Quadrant)

In Unit 2 you learned how to prepare a set of number pairs from an equation, and how to plot the points corresponding to the number pairs. You drew a curve connecting the points and called this curve the graph of the equation. In more advanced mathematics it can be proved that the curve joining the points is the graph of the equation. This means that any number pair which satisfies the equation corresponds to a point on the curve. The converse statement is also true. The number pair (coordinates) for any point on the curve satisfy the equation. In this discussion we are assuming that the graph of $y = 2x$ is a straight line.

Exercises 6-1a

1. Prepare a set of 5 number pairs for $y = 2x$ in the first quadrant. What numbers are suitable for x? Can y be negative?
2. Plot the points whose coordinates you found in Problem 1. Draw a curve through the points. What kind of curve is it?
3. Show that if a is the x-coordinate of a point on the line, then $2a$ is the y-coordinate of this point.
4. Choose a point S on the line and draw a perpendicular to the x-axis from S.
   (a) Show that OA = the x-coordinate of S.
   (b) Show that SA is the y-coordinate of S.

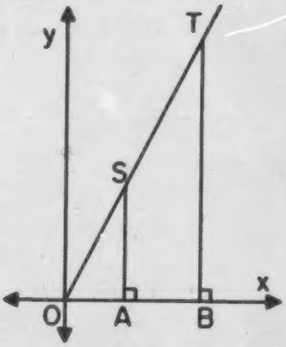


Figure 6-1

(c) Show that $SA = 2(OA)$.

(d) Show that $\frac{SA}{OA} = \frac{x}{y} = 2$.

5. Choose another point T on the graph of $y = 2x$ and draw TB perpendicular to the x-axis.

(a) Let the coordinates of T be (x',y'). Explain why $y' = 2x'$.

(b) Show that $OB = x'$ and that $BT = y'$ or $2x'$.

(c) Show that $\frac{BT}{OB} = \frac{y'}{x'} = 2$.

Note that the coordinates of T were written as (x',y'). It is convenient at times to use the same letters over again and to distinguish them by a special mark. Read x' as x-prime and y' as y-prime. The "prime" is merely part of the name and is not an exponent.

From Problems 4 and 5 we have $\frac{y}{x} = 2$ and $\frac{y'}{x'} = 2$, for $x \quad 0$, $x' \quad 0$. Tell why $\frac{y}{x} = \frac{y'}{x'}$ and $\frac{x}{y} = \frac{x'}{y'}$.

This states that the ratio of two legs of the small right triangle in Figure 6-1 is equal to the ratio of the two corresponding legs of the large right triangle. It is convenient to compare these sides in the following way:

$$\frac{y}{y'} = \frac{x}{x'}$$

but we must prove that we may do this.

We know $\frac{y}{x} = \frac{y'}{x'}$

then $\frac{y}{x} \cdot \frac{x}{y'} = \frac{y'}{x'} \cdot \frac{x}{y'}$

and $\frac{y}{y'} \cdot \frac{x}{x} = \frac{x}{x'} \cdot \frac{y'}{y'}$

Why are these names which look different, names for the same rational numbers? What property of equations is used here?

Hence $\frac{y}{y'} = \frac{x}{x'}$

What property is used here?

Corresponding sides, as you have learned, are the sides which are opposite the angles of equal measure in the two triangles. The sides of length x' and x are corresponding sides. So are the sides of length y' and y. Translated back to the triangles, this equation says that for these two right triangles, the ratios of two pairs of corresponding sides are equal.

Exercises 6-1b

1. Write $\frac{x'}{x} = \frac{y'}{y}$ in terms of OA, SA, OB, and TB.
2. Name the corresponding angles of triangles OAS and OBT.
3. Would it still be true that the ratios of two pairs of corresponding sides are equal if S and T had been different points on the graph of $y = 2x$ in the first quadrant?
4. Suppose the graph had been $y = 65x$ or $y = \frac{x}{10}$, would $\frac{x'}{x} = \frac{y'}{y}$ for any points S and T on this graph?
5. The coordinates (x',y') of a point on $y = 5x$ may be given in terms of x' as $(x',?)$. Replace? by an expression for y' which involves x'.
6. The y-coordinate of a point on $y = \frac{1}{3}x$ may be given in terms of x as $(x,?)$.

6-2. A Special Set of Right Triangles

The sides OS and OT were not included in the discussion in Section 6-1. Let us see what we can find out about these sides. Since we know that $\frac{OA}{OB} = \frac{SA}{TB}$ for 2 pairs of sides, it seems reasonable to ask about the ratio of $\frac{OT}{OS}$, the third pair.

Although measurements are only approximate, we might try measuring OT and OS in our search for a clue concerning what seems likely to be true. It will help us if we draw TB in a special way this time.

In the new figure, OA will be our chosen unit of length and we shall make OA = AB. Incidentally, must OA have the same measure in inches on every paper in your class? The actual length is not important, since we are working with ratios and we know that $\frac{k}{2k}$ is $\frac{1 \times k}{2 \times k}$ which is $\frac{1}{2}$ no matter what k is (excluding k = 0). Measure $\overline{OT}$ and $\overline{OS}$. What do you conclude? Points O, S, and T are on the graph of y = 2x. Draw $\overline{SK}$ perpendicular to $\overline{TB}$. Locate ΔOAS and ΔSKT in this diagram.

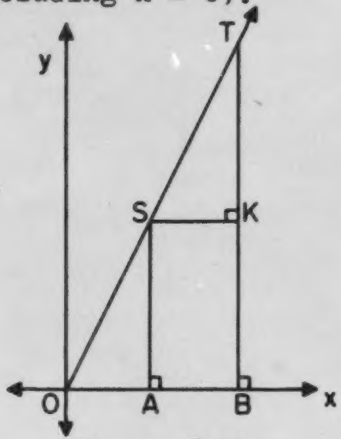


Figure 6-2

Exercises 6-2

1. (a) What can you say about $\overline{AS}$ and $\overline{BT}$? Are they parallel? Why?
   (b) Show that if the coordinates of T are (x',y') then x' = OB and y' = BT.
   (c) Show that BT = 2(OB).
   (d) Show that 2(OB) = 4(OA).
   (e) Express BT in terms of OA.
2. Consider $\overline{OA}$ the unit of measurement and let it be 1.
   (a) What is the measure of $\overline{OB}$?
   (b) What is the measure of $\overline{SA}$?
   (c) What is the measure of $\overline{BT}$?
3. (a) What kind of triangle is ΔOAS? Δ OBT? Why?
   (b) Name the right angle in each of these triangles.
   (c) Name the angle which is in both triangles.

(d) What do you know about the sum of the angles of any triangle?

(e) Show that m($\angle$OSA) = m($\angle$OTB).

4. In what other way, using Problem 1(a) can you show that m($\angle$OSA) = m($\angle$OTB)?

5. (a) What can you say about $\overline{AB}$ and $\overline{SK}$?

(b) What can you say about $\overline{AS}$ and $\overline{BK}$?

(c) What kind of figure is ABKS?

(d) Show that SK = AB.

6. Show that SK = 1.

7. Show that AS = KT. (From Problem 2, BT is 4 and SA is 2.)

8. Since you know OA = SK, AS = KT, and m($\angle$OAS) = m($\angle$SKT) what can you say about ΔOAS and ΔSKT? Can we say corresponding sides and angles are congruent?

9. As a result of Problem 8, OS = ST. Tell why.

10. From OS = ST show that OT = 2(OS) and that $\frac{OT}{OS} = 2$.

11. Write the corresponding sides of triangles OTB and OSA in pairs.

In Exercise 10 you found without relying on measuring that $\frac{OT}{OS} = 2$.

In the first figure, you made OB = 2(OA) so $\frac{OB}{OA} = 2$. From Problem 2 $\frac{BT}{AS} = 2$. Therefore $\frac{OT}{OS} = \frac{OB}{OA} = \frac{BT}{AS}$ since each ratio is 2.

This statement tells us that in the two triangles OAS and OBT, in which corresponding angles are equal, pairs of corresponding sides are proportional.

In the special right triangle in Figure 6-2

(1) the ratio of the longer leg to the shorter leg in each triangle is 2

(2) the lengths of the legs of one triangle are twice the lengths of the legs of the other triangle.

6-3. The Graph of $y = ax$ (First Quadrant, a is a positive real number)

In Sections 1 and 2 the right triangles were in a very special set. They were the right triangles formed by the graph of $y = 2x$ and certain perpendicular lines drawn so that the ratio of the sides on the x-axis is equal to 2.

What happens if we change to the graph of $y = 3x$? or $\frac{7}{3}x$? or where a is any positive real number? The following questions will help you reach some conclusions. Look for properties which seem to be independent of a particular number for a in $y = ax$. Search for patterns. In this discussion we are assuming that the graph of $y = ax$ is a straight line, for "a" any positive real number.

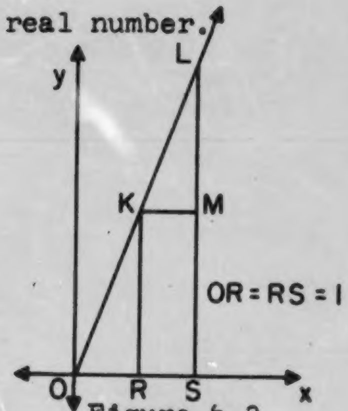


Figure 6-3

Exercises 6-3a

1. (a) Draw the graph of $y = 3x$.
   (b) Select a point K, draw KR perpendicular to the x-axis. Let $OR = 1$.
   (c) What is KR in terms of OR? Write a numeral for KR. Write the coordinates of K.
   (d) Draw $OS = 2$ and then draw LS perpendicular to the x-axis.
   (e) What is LS?
   (f) What is the ratio of $\frac{OS}{OR}$? of $\frac{SL}{RK}$?
2. (a) By measurement find the approximate value of $\frac{OL}{OK}$.
   (b) By means of steps like those in Section 6-2 show that $\frac{OL}{OK} = 2$.

Draw KM perpendicular to SL and use congruent triangles as in Exercises 6-2.

3. (a) In this set of triangles based on the graph $y = 3x$, what is the ratio of the length of the longer leg of a triangle to the length of the shorter leg?
   (b) What was this ratio in the case of the triangles based on $y = 2x$?
   (c) Are you beginning to see a pattern?
   (d) How do the lengths of the legs of one triangle in Figure 6-3 compare with the lengths of the corresponding legs of the other? Note that again OS was chosen to be 2(OR).
4. (a) Graph $y = \frac{7}{3}x$.
   (b) Explain why $m(\angle OAR) = m(\angle OBT)$. Are these angles congruent?
   (c) If OA is 1, what is RA?
   (d) If OB is 2, what is BT?
   (e) State the ratio of $\frac{AR}{OA}$ in numerals.
   (f) State the ratio of $\frac{BT}{OB}$ in numerals.
   (g) Tell why these ratios are equal.
   (h) Does the value of these ratios depend upon the measures of $\overline{OA}$ and $\overline{OB}$? What ratios do?

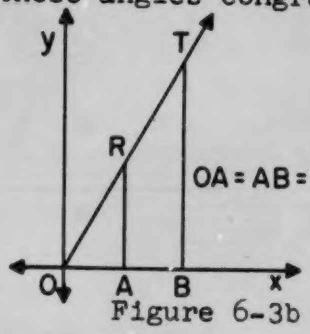
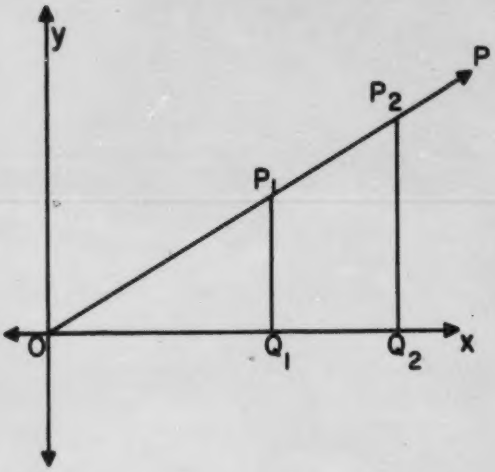


Figure 6-3b

5. Draw a figure like Figure 6-3b, but on a larger scale. Find by measurement and division, the ratios $\frac{BT}{AR}$, $\frac{OB}{OA}$, $\frac{OT}{OR}$. Do these ratios appear to be equal?
6. Since (from Problem 4) $\frac{AR}{OA} = \frac{BT}{OR} = \frac{7}{3}$, $\frac{BT}{AR} = \frac{OB}{OA}$. Why?

Now let us examine the graph of equations like $y \neq ax$ for $a \neq 0$. Here a may be any positive real number. In the special graph $y = 2x$ what was a? What was a in the case of $y = 3x$? in the case of $y = \frac{7}{3}x$?



In the figure at the right the ray $\overrightarrow{OP}$ represents the graph of $y = ax$ in the first quadrant. Do not regard it as drawn to scale.

P_1 is a point on the graph $y = ax$.

P_1Q_1 is perpendicular to the x-axis.

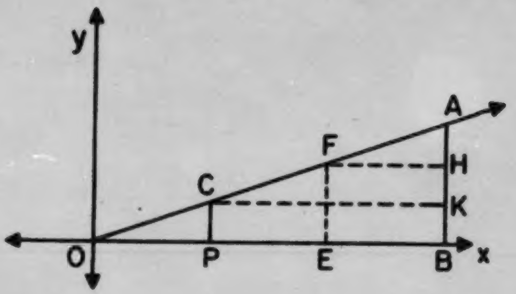
P_2 is a point on $y = ax$.

P_2Q_2 is perpendicular to the x-axis.

Exercises 6-3b

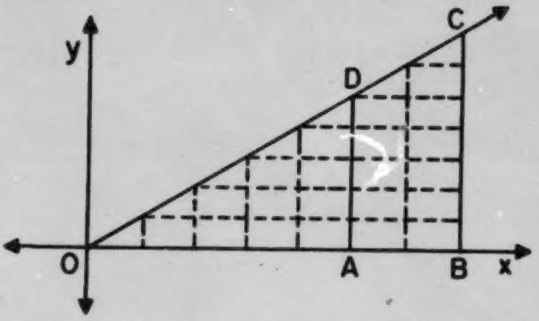
1. If the coordinates of P_1 are (x_1y_1) what is y_1 in terms of a? Note that $y = ax$ and $y_1 = ax_1$ since P_1 is on the graph.
2. Show that $\frac{y_1}{x_1} = \frac{Q_1P_1}{OQ_1}$
3. Show that the ratio of $\frac{Q_1P_1}{O\ Q_1}$ is a.
4. Let the coordinates of P_2 be $(x_2,\ y_2)$ and show that $y_2 = ax_2$.
5. Show that P_2Q_2 is ax_2 and $OQ_2 = x_2$ so that $\frac{P_2Q_2}{OQ_2} = a$.
6. By measurement show that $\frac{OP_2}{OP_1}$ appears to equal $\frac{P_2Q_2}{P_1Q_1}$ and $\frac{OQ_2}{OQ_1}$.
7. Use the equation $y = ax$ to find the ratio of y/x. $(x \neq o)$
8. Use Problems 3,5,7 to show $y/x = \frac{P_1Q_1}{OQ_1} = \frac{P_2Q_2}{OQ_2}$.
9. Use the arguments in Section 6-2 involving congruent triangles to show that $\frac{OP_2}{OP_1} = \frac{P_2Q_2}{P_1Q_1} = \frac{OQ_2}{OQ_1}$. Why is it necessary in this proof to know that $OQ_2 = 2(OQ_1)$? Assume that $OQ_2 = 2(OQ_1)$.

10. In the figure at the right, $\frac{OD}{OB} = \frac{1}{3}$. The dotted lines should suggest a way you can prove certain triangles congruent and thus show that $\frac{OC}{OA} = \frac{1}{3}$. Try it.



11. The ratio $\frac{OA}{OB}$ is $\frac{5}{7}$. Show $\frac{CB}{DA} = \frac{5}{7}$ and $\frac{OD}{OC} = \frac{5}{7}$.

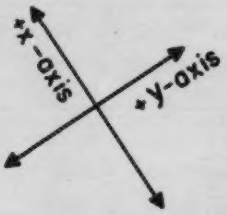
CD and OB should be divided into 7 parts by measurement.

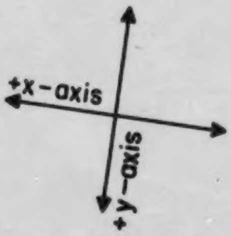


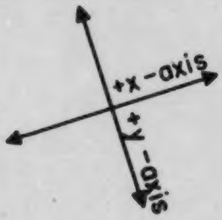
12. Could the methods of Exercises 11 and 12 be used, if $\frac{OA}{OB} = k$, where k is an irrational number?

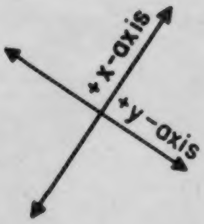
6-4. The Similarity Property for Right Triangles

In our work we have followed certain conventions in setting up coordinate axes, making the x-axis horizontal and the y-axis vertical. We have chosen to regard the right as a positive direction for the x-axis, and upward as a positive direction for the y-axes. The opposite directions in each case were negative. These are merely convenient conventions and changing them is quite possible. Look at the coordinate axes below. Note that all of them retain the perpendicular characteristic, which itself is not always observed.





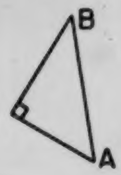


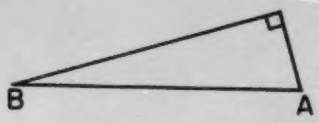


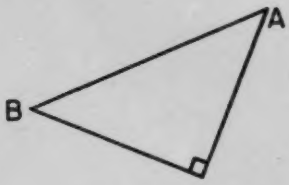
The first quadrant is the one in which both x and y are positive. Point out the first quadrant in each situation above.

In sections 1,2, and 3 we studied right triangles in a special position with reference to a pair of coordinate axes. Actually the properties which we observed in these special cases are true for any right triangles.

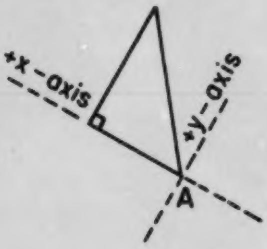
Consider the following right triangles.

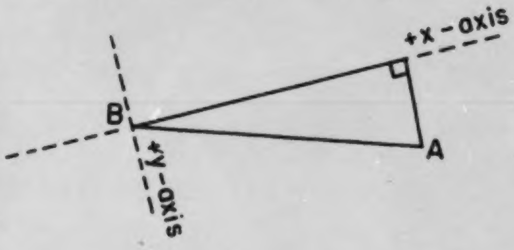


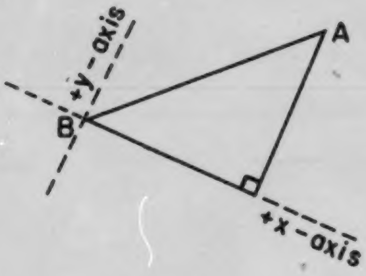




In each case we could choose coordinate axes so that A or B lies on the origin, so that one of the legs of the triangles lies along the positive x-axis, and so that the interior of the triangle is in the first quadrant. This is illustrated in the following figures in which the coordinate axes are represented by dotted lines.



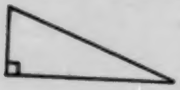




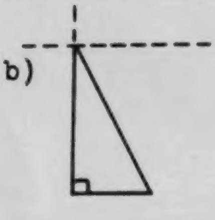
Exercises 6-4a

1. Show by using dotted lines as shown in the illustration, possible placement of coordinate axes in the following figures.

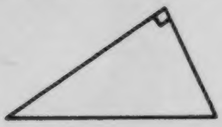
(a)



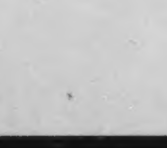
(b)



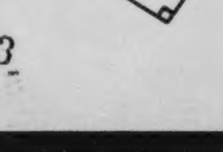
(c)



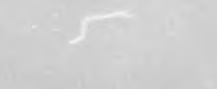
(d)



(e)

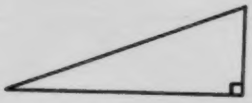


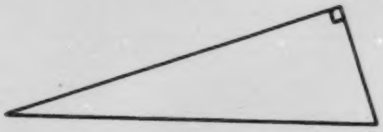
(f)

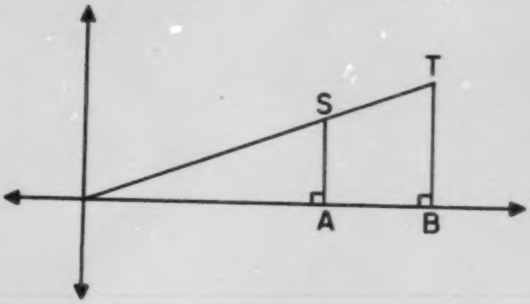


Note that after the axes are located, the hypotenuse of the right triangle is on the line through the origin. The equation of the line is y = ax where 'a' is a particular positive real number not zero, for a particular line. You might find it easier to think of moving the triangles in the plane, by rigid motion, so that they are in the position used in sections 1, 2, and 3. No matter which method is used, all of the arguments used in sections 1, 2, and 3 hold for all right triangles of the sets described.

When you move the triangle as suggested here, two right triangles for which a pair of corresponding acute angles are equal will look like this:

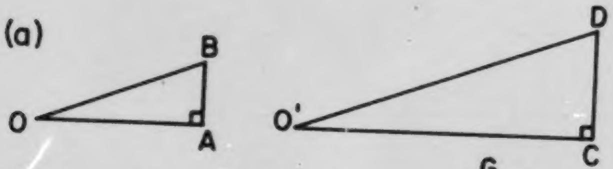


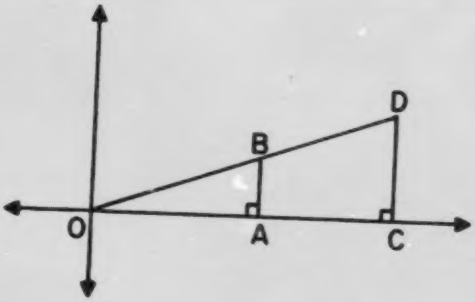


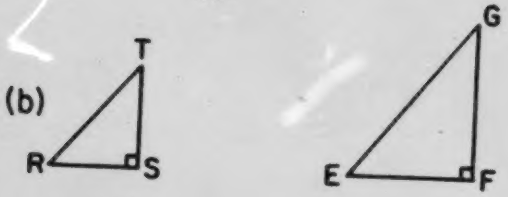


Exercises 6-4b

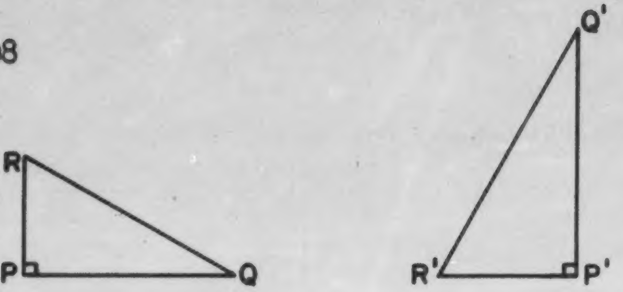
1. Draw the following pairs of triangles on a pair of axes in standard form as in the figure on the right:



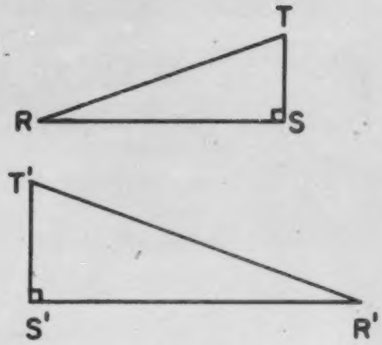




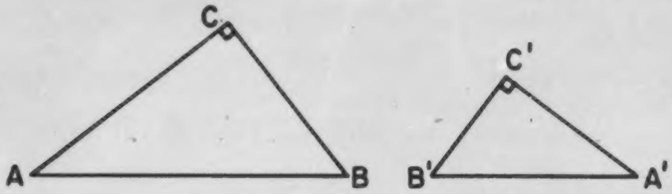
(c)



(e)



(d)



2. If you know the equation of the line on which the hypotenuse of a triangle lies, how can you use it to tell the ratio of the legs of the triangle?

3. The ratio of the sides can be estimated by measuring the lengths of the sides. Find the ratios given below:

(a) $\frac{AB}{DC}$ and $\frac{OA}{OC}$ (c) $\frac{PQ}{P'Q'}$ and $\frac{PR}{P'R'}$ (e) $\frac{TS}{T'S'}$ and $\frac{RT}{R'T'}$

(b) $\frac{RS}{EF}$ and $\frac{RT}{EG}$ (d) $\frac{AC}{A'C'}$ and $\frac{BC}{B'C'}$

Property 1. If a pair of corresponding acute angles of two right triangles are equal, then the ratios of corresponding sides are equal.

We often say that if the corresponding angles of two triangles are equal, then the two triangles are similar. This is the definition which we will use for similar triangles. Property 1 could be stated as:

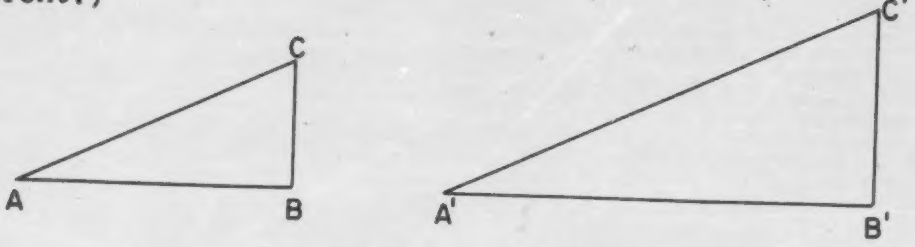
If two triangles are similar then the ratios of corresponding sides are equal.

It should be entirely clear that the deductive arguments used do not apply to all pairs of right triangles. Exercises of section 6-3 suggest a way in which congruent triangles may be used in an informal deductive proof of Property 1 for two right triangles such that the ratio of lengths of corresponding sides is a rational number. Actually the property holds even though the ratio is an irrational number.

In Exercises below you are asked to find the ratio of the lengths of the hypotenuse of two similar right triangles by finding these lengths using the Theorem of Pythagoras. This would provide us a way of verifying Property 1 for all pairs of similar right triangles. However the proof using this method in the general case will be postponed until you have had more experience with numbers like $\sqrt{45}$, $\sqrt{48}$, and $\sqrt{200}$.

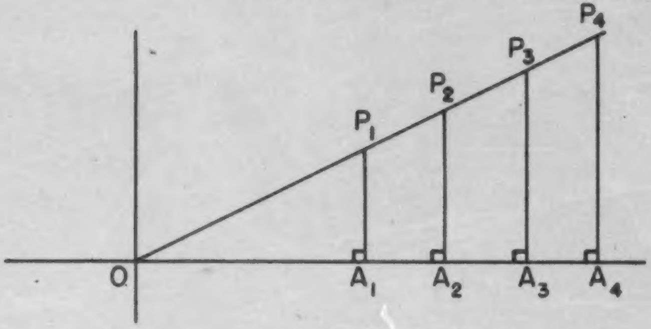
Exercises 6-4c

Use the Theorem of Pythagoras to find the missing sides and then find the 3 ratios of the corresponding sides, (Of course these triangles are not drawn to scale because of the several parts, each different.)



1. If AB = 4, BC = 3, A'B' = 20, B'C' = 15.
2. If AB = 12, BC = 5, A'B' = 24, B'C' = 10.
3. If AB = 15, BC = 8, A'B' = 45, B'C' = 24.
4. If AB = 8, BC = 6, A'B' = 12, B'C' = 9.
5. If AB = 2, BC = 1, A'B' = 4, B'C' = 2.

6-5. Sines and Tangents



$P_1 : (x_1, y_1)$
$P_2 : (x_2, y_2)$
$P_3 : (x_3, y_3)$
$P_4 : (x_4, y_4)$

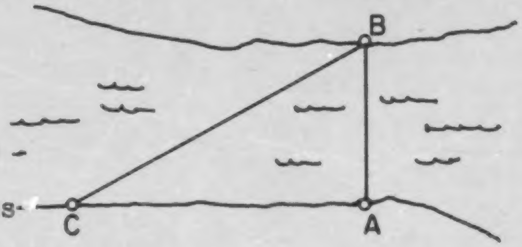
The figure shows a part of the graph of $y = \frac{1}{2}x$ in the first quadrant. Why are the four right triangles shown in the figure similar triangles? Since these triangles are similar, the ratios of corresponding sides in any pair of them are equal. For example,

$$\frac{A_1P_1}{OA_1} = \frac{A_2P_2}{OA_2} = \frac{A_3P_3}{OA_3} = \frac{A_4P_4}{OA_4}\ .$$

We could draw in the figure other right triangles in positions like those in the figure, which are similar to the four triangles. How many of these could be done?

Express the ratios above in terms of the coordinates of the points P_1, P_2, P_3, and P_4.

Measure $\angle$ AOP. What does its measurement in degrees appear to be?



Now suppose we wished to know the distance across the river shown in this figure. A and B represent the positions of two trees on opposite banks of the river. The length of $\overline{AC}$ is 20 feet. By use of an angle measuring device the measurement of $\angle$ ACB has been determined as 30°. How can the distance AB be determined?

One way would be to draw carefully on a fairly large scale a triangle similar to triangle ABC, and then find the ratio of the sides corresponding to AB and AC. Since the $\angle A_1OP_1$ in the figure above has a measurement of 30°, we could use any one of these triangles. From measurement or from the graph, we found $\frac{A_1P_1}{OA_1} = \frac{1}{2}$.

Since triangle ABC in the sketch of the river and trees is similar to triangle OA_1P_1, by matching corresponding sides, we can say,

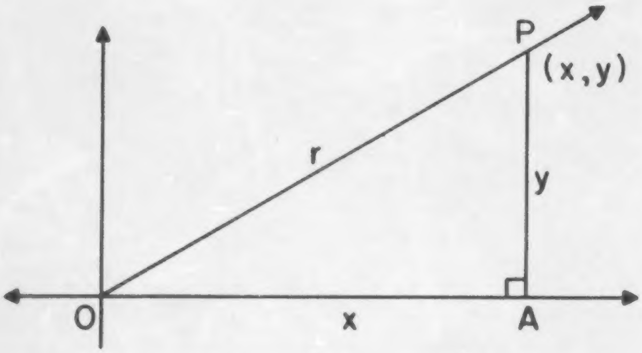
$$\frac{A_1P_1}{OA_1} = \frac{1}{2} = \frac{AB}{CA}.$$

Since AC = 20, AB = 40. The width of the river is 40 feet.

We have used similar triangles to make an indirect measurement of the distance across the river.

Because this is a useful and fairly common application of properties of similar right triangles, it has been found to be convenient to give special names to the ratios of the sides of triangles. Actually there are six ratios of corresponding sides of a triangle. In this section we will give our attention to two of them. These two ratios are, $\frac{A_1P_1}{OA_1}$ and $\frac{A_1P_1}{OP_1}$.

List the other four ratios of sides of the triangle OA_1P_1. (Don't forget $\frac{OA_1}{A_1P_1}$ which is different from $\frac{A_1P_1}{OA_1}$.) Perhaps you would prefer to list these ratios using x_1, y_1 and OP_1. The two ratios then become $\frac{y_1}{x_1}$ and $\frac{y_1}{OP_1}$.



Consider a coordinate system for the plane, and $\angle AOP$. The segment $\overline{OA}$ lies on the positive x-axis and the segment $\overline{OP}$ lies on the ray $\overrightarrow{OP}$. If the coordinates of P are (x,y) then

$$OA = x$$

$$AP = y$$

Let r = OP. Then, by Pythagorean property

$$x^2 + y^2 = r^2$$

$$r = \sqrt{x^2 + y^2}$$

We shall call the ratio, $\frac{y}{r}$, the sine of angle AOP, and the ratio, $\frac{y}{x}$, the tangent of angle AOP. In a shorter way, we shall write

$$\sin \angle AOP = \frac{y}{r} \qquad \tan \angle AOP = \frac{y}{x}$$

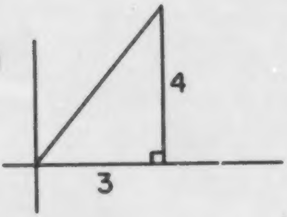
These expressions are read, "the sine of angle AOP equals $\frac{y}{r}$"., and "the tangent of angle AOP equals $\frac{y}{x}$".

The ratios are sometimes called the trigonometric ratios. The word "trigonometry" is simply "trigon" and "metry", and is suggested by the phrase, the measurement of trigons (triangles).

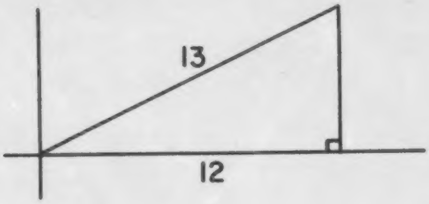
Exercises 6-5

1. Find the sine and the tangent of the angles, with vertex at the origin, shown in the following figures:

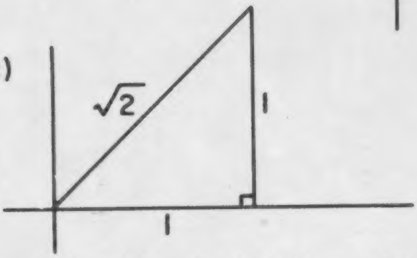
(a)



(b)



(c)



What is the measurement of the angle at the origin in (c)?

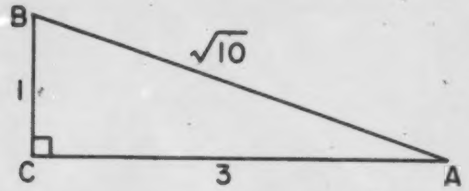
2. Draw a right triangle in the appropriate position with reference to a pair of coordinate axes, for which one endpoint of the hypotenuse is (0,0) and the other is at P:

   (a) P: (4,3)   (b) P: (2,5)   (c) $(1, \sqrt{3})$.

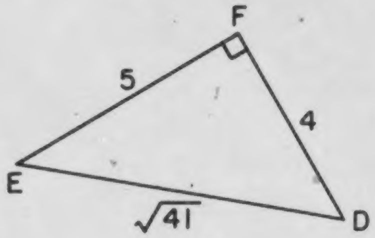
3. List the sines and tangents of angles with vertex at (0,0) in the triangles you have drawn in Exercise 3.

4. Place the following triangles in appropriate positions with reference to coordinate axes and find the sines and tangents of both acute angles in each:

(a)



(b)



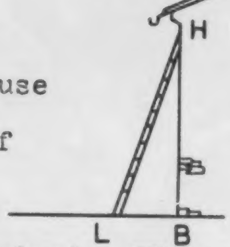
5. Draw on metric coordinate paper, using a protractor, the following angles, and by measurement find the sine and tangent of each.

   (a) 40°   (b) 70°

   In finding these ratios by measurement use the base of the triangle as 10 centimeters.

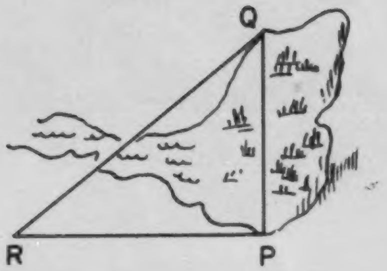
6. Use the values for sin 40°, tan 40°, sin 70°, tan 70°, as needed in solving the following:

   (a) A ladder, 15 feet long, leans against a house as shown in the figure. The measurement of ∠BLH is 70 degrees. Find BH.



   (b) The distance across a swamp is to be determined. RP has been found to be 42 feet. Angle PRQ has a measurement of 40 degrees. Find PQ.

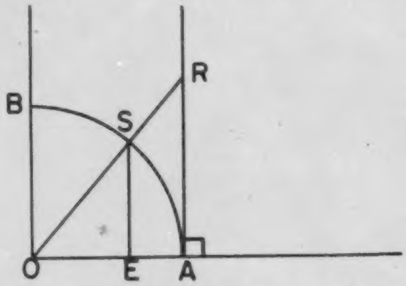
Since a small stream crosses the path represented by RQ, it is difficult to determine this distance by direct measurement. Find RQ.



7. Draw one-fourth of a circle on metric coordinate paper, letting the radius of the circle be 10 centimeters. A sketch is given as a guide. Let the unit of length be OA = 1. $\overline{AT}$ is on a line perpendicular to $\overline{OA}$ at A.

$$\tan \angle AOR = \frac{AR}{OA} = \frac{AR}{1} = AR.$$

The values of the tangents of angles can be read directly from $\overline{AR}$ in your diagram. With a protractor draw the angles of the following measurements and give the values of the tangents of the angle from the coordinate paper.



(a) 30° (b) 38° (c) 50° (d) 62°

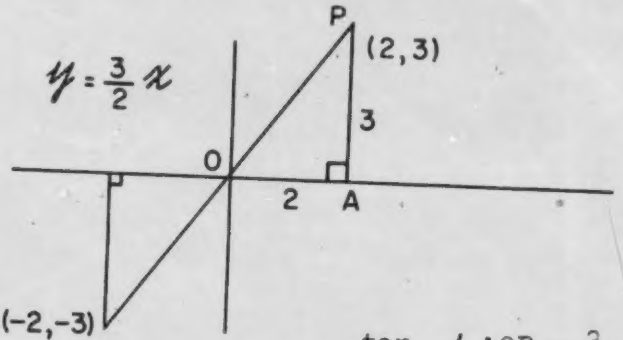
8. In the graph prepared for Exercise 7, measure $\angle AOT$. What is the measurement of the angle for which the tangent is 2? Mark the intersection of the line $\overleftrightarrow{AR}$ with the top of the coordinate chart as T.

9. From the graph prepared for Exercise 7, find the sines of the angles given in Exercise 7.

(Hint: Read from the graph paper the lengths of segments corresponding to ES for the various angles.)

6-6. Slope of a Line

If we refer again to graphs $y = ax$ like those used in sections 6-1, 6-2, and 6-3, we can see a special application of the tangent of an angle. The figure represents the graph of $y = \frac{3}{2}x$ (but no longer restricted to the first quadrant).

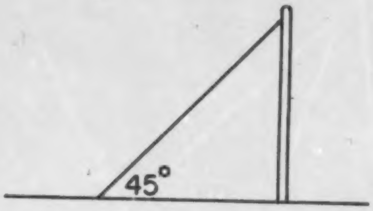


$\tan \angle AOP = \frac{3}{2}$

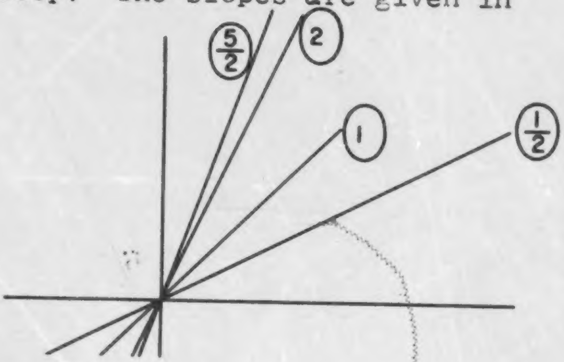
$\frac{3}{2}$ is called the slope of the line $y = \frac{3}{2}x$

The word "slope" suggests a measure of the steepness of the line. The slope of a line represented by a road up a steep hill will be larger than the slope of a line represented by a road with only a slight rise. Do you see why?

In the figure the line represented by the supporting or guy wire for a pole, has a slop of 1. Why?



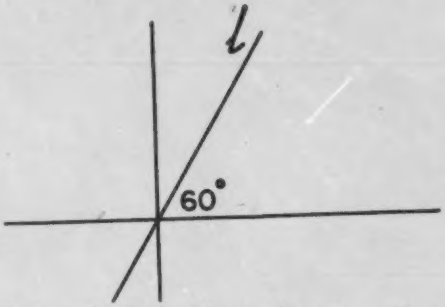
In the figure, lines of slopes $\frac{1}{2}$, 1, 2, and $\frac{5}{2}$, are drawn on one set of axes. You will notice that the slope of the line becomes greater as the lines become more steep. The slopes are given in the small encircled numerals. Would it be reasonable to say that line of slope 2 is twice as steep as the line of slope 1? Why not?



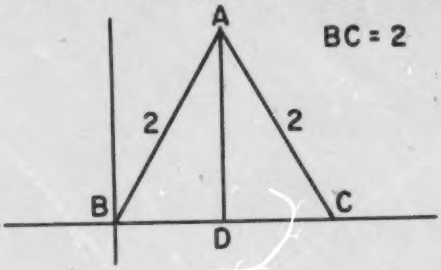
(There are lines for which the slope is negative but we shall not consider lines like this in this section).

What might be given as the slope of the x-axis? the y-axis?

Suppose line ℓ passes through the origin as shown in the figure. The smaller angle formed by ℓ and the x-axis has a measurement of 60 degrees. What is the slope of ℓ? We could choose a point P on ℓ, determine its coordinates approximately by measurement, and then compute the ratio, giving the slope. Draw a fairly large figure and try this method.



Since the measure of the equal angles of an equilateral triangle is 60 degrees, we could also find the slope of ℓ by considering some of the properties of an equilateral triangle.



In the equilateral triangle ABC, AD is an altitude of the triangle. By use of ruler and protractor, measure the sides of triangles BDA and CDA. Do the triangles appear to be congruent?

Draw an equilateral triangle each side of which is three inches in length. Draw one altitude of the triangle. Cut the region which is the interior of the triangle out of paper. Fold the region along the altitude. What appears to be true of the two parts of this interior?

Referring to the figure above we can show that the triangles BDA and CDA are congruent. Both of these triangles contain the side BD and both are right triangles.

$$m(\angle ABC) = m(\angle BCA) \qquad \text{Why?}$$

Since the measures of corresponding angles of the two triangles are equal and since they have one side in common, the two triangles are congruent.

We conclude $\quad BD = DC$

and $BD = 1$, $\quad$ since $BC = 2$.

The measurement of $\angle DBA$ is 60 degrees.

By the theorem of Pythagoras $\quad (BD)^2 + (AD)^2 = (AB)^2$

$$(1)^2 + (AD)^2 = (2)^2$$

$$1 + (AD)^2 = 4$$

$$(AD)^2 = 3$$

$$AD = \sqrt{3}$$

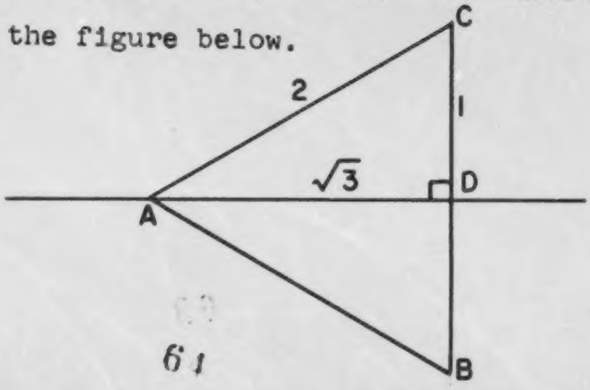
In the figure the coordinates of A are $(1, \sqrt{3})$. Why?

The slope of the line on which $\overline{BA}$ lies is $\sqrt{3}$. This is also the slope of ℓ in the figure above since $\overline{BA}$ extended would represent .

$$\tan 60^\circ = \sqrt{3}$$

Find the sin 60°.

Place an equilateral triangle ABC on the coordinate axis so that it will look like the figure below.



Find the sine and tangent of 30 degrees.

$\sin 30^\circ = ?$ $\tan 30^\circ = ?$

Exercises 6-6

1. Graph the lines $y = \frac{2}{3}x$, $y = \frac{1}{4}x$, and $y = 4x$ on the same set of axes. Use four values of x for each graph. Find the slopes of these lines.

2. Find the slopes of the lines joining the following pairs of points:

   (a) (0,0) and (1,3)
   (b) (0,0) and (2,3)
   (c) (0,0) and $(\frac{1}{2},5)$
   (d) $(\frac{1}{2},2)$ and (2,8)

3. Draw a rectangle with one vertex at (0,0) and its interior completely in the first quadrant. The lengths of the sides of the rectangle are 3 and 5. Find the slope of its diagonal which lies on (0,0). (There are two answers.)

4. A road rises 10 feet over a distance of one mile. What is the slope of the road?

5. Draw a right isoscles triangle and determine the sine and tangent of angles of measurement, 45 degrees.

6. Find the ratios: (a) $\frac{\sin 60^\circ}{\sin 30^\circ}$, (b) $\frac{\tan 60^\circ}{\tan 30^\circ}$

   (c) How is the answer to this problem related to the conclusion that a line of slope 2 is not twice as steep as a line of slope 1.

7. Choose an appropriate scale on coordinate axes so that you can compare the graphs of $y = 7x$ and $y = 8x$. How does the measure of the angle determined by these lines compare with the measure of the angle formed by lines of slope 1 and slope 2?

6-7. Reading a Table

In the Exercises of the last section we found approximate values of sines and tangents of angles by measurement, and by using so-called line values as found from a graph. Scientists use tables of values of the trigonometric ratios computed to a given number of decimal places, according to the particular application for which the values are needed. Advanced mathematical methods give much greater accuracy for table readings than could be found by measurement.

On the next page, values of sines and tangents are given correct to four decimal places for sines and tangents of angles with measurement given in degrees, to the nearest degree. These will provide a sufficient degree of accuracy for our purpose.

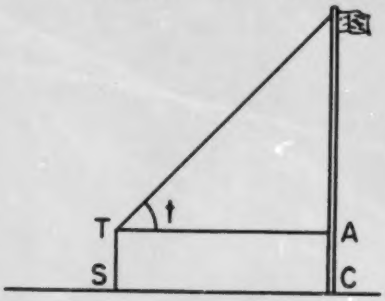
Reading the table we find the following approximate values of some of these ratios.

$\sin 22^\circ \approx .3746$ $\qquad$ $\sin 57^\circ \approx .8387$

$\tan 22^\circ \approx .4040$ $\qquad$ $\tan 73^\circ \approx 3.2709$

Do you see how these values were obtained from the tables? If the angle is greater in measurement than 45 degrees the angle is given in the right hand column and the tables are read from the bottom of the page.

Suppose we want to know the height of a flagpole, represented in the figure. A man whose eye-level is 6 feet above the ground, walks 20 feet in a straight line perpendicular to the base of the flagpole and with an instrument for measuring angles (a transit, perhaps) and measures the angle marked, t.



TRIGONOMETRIC RATIOS

Angle	Sine	Tangent	Cotangent	Cosine	
0°	.0000	.0000	....	1.0000	90°
1°	.0175	.0175	57.290	.9998	89°
2°	.0349	.0349	28.636	.9994	88°
3°	.0523	.0524	19.081	.9986	87°
4°	.0698	.0699	14.301	.9976	86°
5°	.0872	.0875	11.430	.9962	85°
6°	.1045	.1051	9.5144	.9945	84°
7°	.1219	.1228	8.1443	.9925	83°
8°	.1392	.1405	7.1154	.9903	82°
9°	.1564	.1584	6.3138	.9877	81°
10°	.1736	.1763	5.6713	.9848	80°
11°	.1908	.1944	5.1446	.9816	79°
12°	.2079	.2126	4.7046	.9781	78°
13°	.2250	.2309	4.3315	.9744	77°
14°	.2419	.2493	4.0108	.9703	76°
15°	.2588	.2679	3.7321	.9659	75°
16°	.2756	.2867	3.4874	.9613	74°
17°	.2924	.3057	3.2709	.9563	73°
18°	.3090	.3249	3.0777	.9511	72°
19°	.3256	.3443	2.9042	.9455	71°
20°	.3420	.3640	2.7475	.9397	70°
21°	.3584	.3839	2.6051	.9336	69°
22°	.3746	.4040	2.4751	.9272	68°
23°	.3907	.4245	2.3559	.9205	67°
24°	.4067	.4452	2.2460	.9135	66°
25°	.4226	.4663	2.1445	.9063	65°
26°	.4384	.4877	2.0503	.8988	64°
27°	.4540	.5095	1.9626	.8910	63°
28°	.4695	.5317	1.8807	.8829	62°
29°	.4848	.5543	1.8040	.8746	61°
30°	.5000	.5774	1.7321	.8660	60°
31°	.5150	.6009	1.6643	.8572	59°
32°	.5299	.6249	1.6003	.8480	58°
33°	.5446	.6494	1.5399	.8387	57°
34°	.5592	.6745	1.4826	.8290	56°
35°	.5736	.7002	1.4281	.8192	55°
36°	.5878	.7265	1.3764	.8090	54°
37°	.6018	.7536	1.3270	.7986	53°
38°	.6157	.7813	1.2799	.7880	52°
39°	.6293	.8098	1.2349	.7771	51°
40°	.6428	.8391	1.1918	.7660	50°
41°	.6561	.8693	1.1504	.7547	49°
42°	.6691	.9004	1.1106	.7431	48°
43°	.6820	.9325	1.0724	.7314	47°
44°	.6947	.9657	1.0355	.7193	46°
45°	.7071	1.0000	1.0000	.7071	45°
	Cosine	Cotangent	Tangent	Sine	Angle

He records his measurement as 40°. What is the height of the flagpole? In the figure,

$$\tan 40^\circ = \frac{AB}{20}$$

$$20 \tan 40^\circ = AB$$

$$20(.8391) \approx AB$$

$$16.782 \approx AB$$

The height of the flagpole is 23 feet to the nearest foot.

Exercises 6-7

1. Use the table to find the following

(a) sin 10°	(f) tan 40°
(b) tan 10°	(g) tan 50°
(c) sin 41°	(h) tan 60°
(d) sin 63°	(i) tan 70°
(e) sin 82°	(j) sin 88°

2. Check the properties of the numbers in the sine columns of the table by a study of the Table. Do you agree with these statements?

   A. The sine of angles in the Table is always between 0 and 1.

   B. The sine of angles increases with the size of the angle between 0° and 90°.

   C. The sine of angles less than 30° is less than $\frac{1}{2}$.

   D. The differences between consecutive Table readings varies throughout the Table.

   E. The difference between the sines of two consecutive angles is greater for smaller consecutive than for large consecutive angles.

3. State properties for the tangent columns which are similar

to those given in Problem 2 for the sine columns.

4. Why is there no value listed in the Table for the tan 90°?

5. Find the following products:

   (a) 100 sin 32° (c) 0.27 sin 73°

   (b) 81 tan 48° (d) 0.05 tan 80°

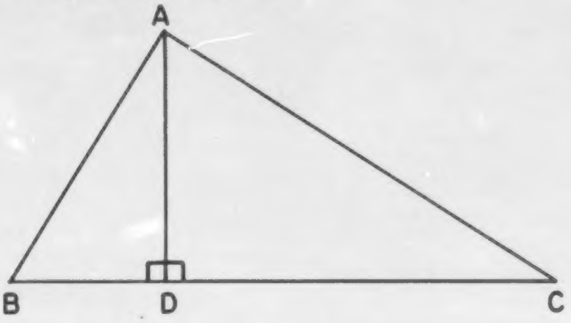
*6. Find (a) AD

   (b) BC

   Given the measurement of

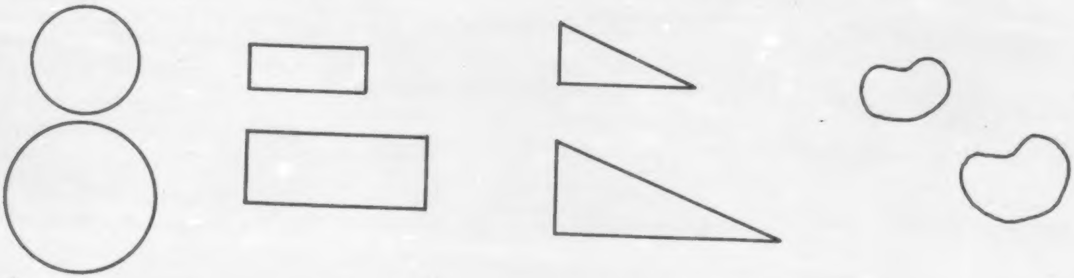
   $\angle ABC$ as 60° and $\angle ACB$ as 32°

   AB = 100



6-8. Proportionality in Geometric Figures

In the figures below there are two circles, two rectangles, two triangles, and two other simple closed curves. In pairs the two figures of a set are said to be similar. They have exactly the same shapes, but their interiors do not have the same size.



It appears that elements of the following sets may be said to be similar

(a) all circles

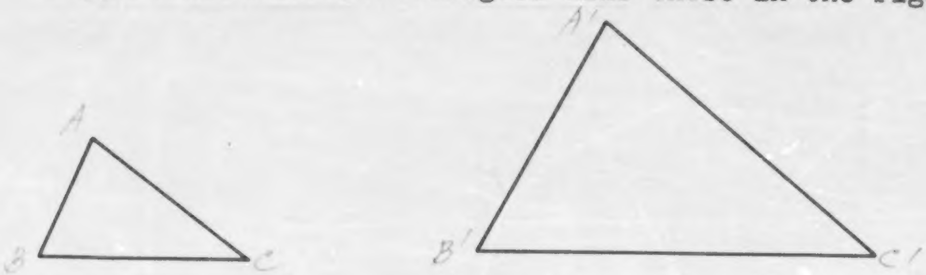
(b) all equilateral triangles

(c) all rectangles of which the ratio of the two adjacent sides is 3

(d) all isosceles right triangles.

Name some other examples of sets of geometric figures of which any two pairs are similar.

Suppose we have two triangles like those in the figures below.



Their angles are congruent in pairs and ratios of sides opposite equal angles are 2 (if the larger side is named first in the ratio). We may use the notation

$$ABC \longleftrightarrow A'B'C'$$

to "match" the triangles. This is a correspondence, but of course it is not a congruence (See Unit 4) since the sides of A'B'C' are larger than the corresponding sides of ABC. Correspondence of this kind is called similarity. In this correspondence

$$A \longleftrightarrow A' \qquad B \longleftrightarrow B' \qquad C \longleftrightarrow C'$$

$$AB \longleftrightarrow A'B' \qquad AC \longleftrightarrow A'C' \qquad BC \longleftrightarrow B'C'$$

and the angle contained in ABC with vertex at A corresponds to the angle contained in A'B'C' with vertex at A'. A similar correspondence is established for the pair of angles with vertices on B and B' and C and C'.

In this correspondence the lengths of the sides of the two triangles form two sequences of positive numbers in a very special relation

$$a' = 2a \qquad b' = 2b \qquad c' = 2c$$

$$\text{or} \qquad \frac{a'}{a} = 2 \qquad \frac{b'}{b} = 2 \qquad \frac{c'}{c} = 2$$

Sequences of positive numbers related in this way are called proportional. Two sequences of numbers which are proportional are sometimes designated as follows:

$$a, b, c, \ldots \sim a', b', c', \ldots$$

The constant ratio, namely

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \ldots$$

is called the constant of proportionality.

Exercises 6-8

1. In which of the following sets are any pair of elements similar?
   - (a) isosceles triangles
   - (b) segments
   - (c) rectangles for which the ratio of length to width is 7
   - (d) parallelogram with angles of 120° and 60°
   - (e) cubes
   - *(f) ellipses
2. Is a congruency (relation between two congruent triangles) a similarity? Why? What is the ratio of corresponding sides?
3. The measure of a side of an equilateral triangle is 4. The measure of a side of another equilateral triangle is $\sqrt{5}$.

   (a) are the triangles congruent? (b) are they similar?
4. Let P_1, P_2, P_3, and P_4 be points on the graph of $y = \frac{5}{2}x$ in the first quadrant. Draw perpendiculars from these points to the x- axis.
   - (a) Is there a correspondence which is a similarity among pairs of these right triangles you formed?
   - (b) If the intersection of the x-axis and the perpendicular on P_1 is A_1, and similarly A_2 is a point on the x-axis

and on the perpendicular on P_2, list corresponding vertices and sides of the two triangles.

5. If a, b, c, ~ a', b', c'

   and if a = 2, b = 3, c = 3, a' = 4, find b' and c'.

6. Draw two triangles of sides a, b, b, and a', b', b', for values given in Problem 5. Measure their angles. Do the angles appear to be equal?

7. $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{d'}{d} = \frac{7}{4}$

   a' = 2, b = 3, c' = 4, d' = 5. Find a, b', c, and d.

8. Two sequences are proportional. The constant of proportionality is 3. In one sequence the first number is 10 and each number of this sequence is 10% greater than the one which precedes it. Find 4 numbers of each sequence.

6-9. The Similarity Property for Triangles

In earlier sections we have considered only pairs of right triangles which are similar. By measuring, by reference to related graphs, and occasionally by informal deduction we have concluded that if an acute angle of one right triangle has the same measure as an acute angle of another right triangle, then pairs of corresponding sides of the two right triangles are proportional.

The converse of this property is also true but no attempt has been made to provide an argument to support the converse. It is suggested as a BRAINBUSTER in this section. In later work in geometry this property will be proved.

Let us now consider similar triangles which are not necessarily right triangles.

Definition. If corresponding angles of two triangles have the same measure then pairs of corresponding sides of the two triangles are proportional.

We will make two assumptions and use these as properties in solving problems.

Assumption 1. If two triangles are similar, then corresponding sides are proportional.

Assumption 2. If a correspondence between pairs of sides of two triangles can be established so that corresponding sides are proportional then corresponding angles have the same measure.

Assumption 2 is a converse of Assumption 1.

Arguments to support Assumption 1 somewhat like those used in developing the property of similar right triangles can be developed but they involve some difficulties which might better be left for later study of geometry. Work with the exercises should make these assumptions appear quite reasonable.

Example. Triangles ABC and A'B'C' are similar. AB = 6, AC = 5, BC = 4, and A'C' = 8. Find A'B' and B'C'.

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} \qquad \frac{B'C'}{BC} = \frac{A'C'}{AC}$$

$$\frac{A'B'}{6} = \frac{8}{5} \qquad \frac{B'C'}{4} = \frac{8}{5}$$

$$A'B' = 6\left(\frac{8}{5}\right) \qquad B'C' = 4\left(\frac{8}{5}\right)$$

$$A'B' = \frac{48}{5} \text{ or } 9\frac{3}{5} \qquad B'C' = \frac{32}{5} \text{ or } 6\frac{2}{5}$$

Sketch the triangles to see if these results appear to be reasonable.

Exercises 6-9

1. Triangles RST and R'S'T' are similar.

   r = ST, r' = S'T', s = RT, and so on.

   Find the lengths of the unknown sides if

   (a) $r = 1$, $s = 3/2$, $t = 2$, $r' = 3$

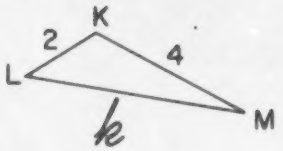
   (b) $r = 3$, $s = 4$, $t = 5$, $t' = 25$

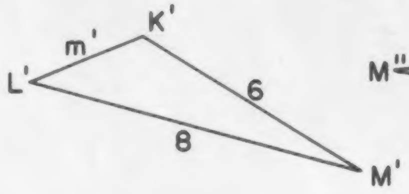
   (c) $r = s = t = 80$, $s' = 30$

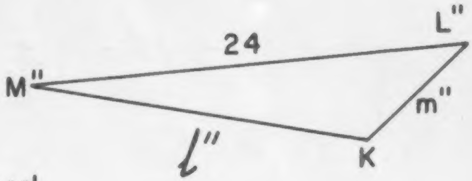
   (d) $r' = s' = 7$, $t = 7$, $t' = 7/2$

   (e) $r' = \sqrt{3}$, $s' = 1$, $t' = 2$, $s = 5$

2. Assuming the three triangles are similar, find the unknown sides:

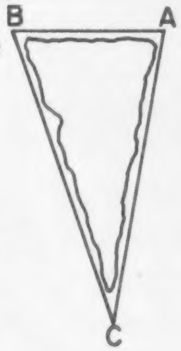






3. A triangular lot is in the shape of an isosceles triangle. The measure of the equal sides is 100 feet and the measure of the third side is 50 feet. A building is to be built on the lot in a shape similar to the lot. If the ratio of lengths of corresponding sides is 1.5 to 1, find the measures of the building.

4. A body of water is roughly triangular in shape as shown in the sketch. Find the length of the road around the lake by drawing a triangle similar to the one given and measure its sides to the nearest centimeter. Let B'C' in your similar triangle be 10 centimeters. The part of the road represented by $\overline{BC}$ is 7 miles long.



The measurement of angle BCA is 30° and of angle ABC is 70°. (The roads represented by $\overline{AB}$ and $\overline{AC}$ are difficult to measure directly because of the terrain.)

5. The lengths of the sides of triangle ABC are 5 inches, 6 inches, and 7 inches. The shortest side of a similar triangle A'B'C' is 10 inches.

   (a) Find the ratio of the perimeters of the triangles.

   (b) Find the ratio of the areas of the triangles.

6. Find the ratios of the perimeters and the areas of the triangles in Problems 1(a) and 1(b).

7. Make a statement about the ratios of the perimeters of two similar triangles.

8. Make a statement about the ratios of the areas of two similar triangles. (Try other examples if no relation appears from Problems 6 and 7.)

9. BRAINBUSTER. Prove that if pairs of corresponding sides of two right triangles are proportional, then the two triangles are similar.

UNIT 7

VARIATION

7-1. Direct Variation

When you were very young, you soon learned that the more candy bars you bought, the more you had to pay. You knew that the more friends you shared a candy bar with, the less each person had. You understood this practical application of variation very clearly, even though you did not think of the mathematics involved.

Now you are ready to think about variation in a somewhat more mature way. From the word, variation, you may expect that this chapter will deal with the ways that quantities change, and how certain changes influence others.

All around you there are quantities that change in a related way. Have you ever watched the changes in the numerals which appeared in the windows of a gasoline pump as the tank of your family automobile was being filled? As you saw the number of gallons increasing, what was happening to the cost?

In mathematics class we say that the cost varies as the number of gallons. If the gasoline you buy is marked to sell for 32 cents per gallon, how can you show how much 5 gallons cost? Complete a table like the one below:

Number of Gallons	1	2	3	4	5	6	7	8	9	10	11
Cost in Dollars	.32										

Observe the computation you carried out and tell in words what the cost is, in terms of the number of gallons and $ .32 per gallon.

Exercises 7-1a

1. Write a sentence in mathematical terms about the total cost t cents of n gallons of gasoline at 32 cents per gallon. In this statement the cost may also be stated as d dollars. Write the sentence a second way, using d dollars. In the first form, what happens to t when n is multipled by 5 ?
2. Suppose the gasoline you bought cost 33.9 cents a gallon. Write a sentence showing the cost, c cents of g gallons of this gasoline. What is true of g when c is doubled?
3. In $c = 5n$, c cenrs is the total cost of n candy bars at 5 cents each. Suppose the price of each bar is raised to 7 cents, and write the formula for the cost of n bars at the higher rate. Is it possible in $c = 5n$, for n to remain unchanged as c increases?
4. If your pace is normally about 2 feet, how far will you walk in n steps? Use d feet for the total distance and write the formula. If n increases, can d decrease at the same time?
5. Write a formula for the number of inches i in f feet. As f decreases what happens to i ?
6. Write a formula for the perimeter, p units, of a square of side s units. As p decreases must s also decrease?

In each exercise above, the equation you wrote followed a pattern. Did you observe that in each formula you had a numeral and two letters? The letters can be replaced by appropriate numerals, but the stated number in each case stays the same. Because this number does not change in a specific formula, it is called a constant. In our statements about direct variation, we shall use k for this constant. For the two variables we shall use x and y. Each formula stated, in effect, that $y = kx$ where k is a constant not zero; k is called the constant of proportionality.

The mathematical sentence $y = kx$ states that y varies directly as x. Other expressions used to describe this relationship are "y varies as x", and "y is proportional to x". If y varies directly as x, then x varies directly as y.

Definition. If two quantities x and y are connected by an equation $y = kx$ where k is a constant not zero, then y varies directly as x.

Exercises 7-1b

1. State the value of the constant, k, in each of the equations you wrote for Exercises 7-1a.
2. Can you write the equation in Problem 6 of Exercises 7-1a in the form $p/s = 4$? What restriction does this form place on s?
3. Find k if y varies directly as x, and y is 6 when x is 2.
4. Find k if y varies directly as x, and y is -3 when x is -12.

5. Sometimes it is required to write an equation from a given set of values. From the information in the table, does it appear that a varies directly as b ? Why? What equation appears to relate a and b ?

a	b
20	2
50	5
-70	-7
100	10
-110	-11

6. From the information in the table below, does it appear that a is proportional to b ? Why? Write an equation if you can.

a	0	2	4	6	8	10
b	-1	0	1	2	3	4

7. Suppose that d varies directly as t and that when t is 6, d is 240. Write the equation relating d and t .

8. Use the relation $y = \frac{3x}{2}$ to supply the missing values in the following ordered pairs: (-4,); (-3,); (-2,); (-1,); (0,); (2,); (5,).

9. Plot the points you found in Problem 8 on graph paper.

10. What seems to be true about the points you plotted in Problem 9? What kind of curve do you think the curve through the points will be? Draw the graph of $y = \frac{3x}{2}$.

11. How many points do you really need in order to draw the graph of $y = \frac{3x}{2}$? Usually it is wise to find an extra point. Why?

12. In the relation of Problem 8, when x is doubled, is y doubled? When x is halved, what happens to y ? When y is multiplied by 10 what happens to x ? Are your statements true for negative values of x and y ?

13. In the general equation $y = kx$ what happens to x if y is halved? What happens to y if x is tripled?

14. Draw the graphs of the equations below using the same set of axes:

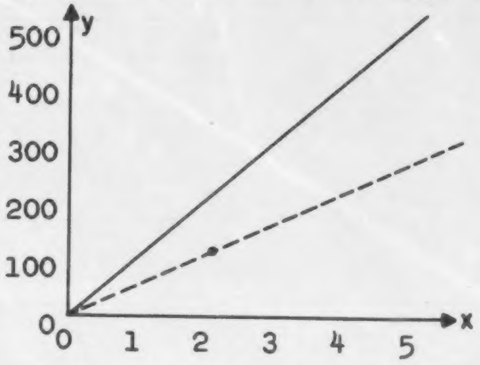
(a) $y = x$
(b) $y = 2x$
(c) $y = \frac{1}{2}x$
(d) $y = 8x$

15. What point do all the graphs in Problem 14 have in common? What is the intersection set of all the graphs? What do you observe about the effect of the change in the constant upon the graph?

16. Without drawing them, what can you say about the graphs of $y = 100x$ and $y = 5x$?

17. Give an example of direct variation when the constant of proportionality is a large number; give another example where $1 > k > 0$.

18. What is the constant of proportionality in $c = 2\pi r$? In the case of $y = 100x$ the values of y are so large that it may be advisable to adjust the scales used on the reference lines in order to draw the graph conveniently. Note the effect this alteration has on the appearance of the graph. Can you understand why it is important to mark the scales, and to observe the scales in studying a graph? Write the equation of the dotted graph. One way to find it is to use a proportion:



$$\frac{y}{x} = \frac{100}{2}$$

$$y = 50x$$

Notice that in the proportion form, x may not be zero. The form $y = 50x$ where x and y may be zero is the equation of the graph.

In the work you have just had with direct variation, the constant of proportionality has been positive in every example. The applications you are likely to meet in junior high school will almost certainly involve positive values for k . Situations exist where the constant of proportionality is negative and you may meet examples as you advance in your study of science. What happens in $y = kx$ when k is negative? When x is multiplied by 10 is y also multiplied by 10? When x increases what can you say about y ? How is this different from the situation for $k > 0$?

7-2. Inverse Variation

The child who shares his apple with a playmate knows that when two people share, each gets only half of an apple. Those of you who have experimented with a lever know that if a weight is tripled, its distance from the fulcrum must be divided by 3 to retain the balance. If you drive 100 miles at 50 miles an hour, the trip requires 4 times as long for you as it does for the airplane which travels at 200 miles an hour.

In direct variation, $y = kx$ (where $k > 0$), you observed that if x increases, y also increases, and if x decreases, y decreases. Perhaps you have anticipated how inverse variation is defined. You may have realized, in the first statement above, that when the number of children increases, the amount of apple available for each one decreases. The following exercises will suggest more examples of inverse variation for you to examine.

Exercises 7-2a

1. (a) The table below, as it is now filled in, shows two possible ways in which a distance of 100 miles can be traveled. Copy and complete the table.

Rate (mi. per hr.)	10	20	25	40	50	60	75	80	100
Time (hours)	10	5							

(b) From part (a), use r for the number of miles per hour and t for the number of hours and write an equation connecting r and t and 100.

(c) When the rate is doubled what is the effect upon the time?

(d) When t increases what happens to r ?

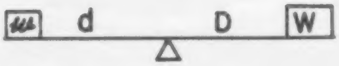
2. (a) Suppose you have 240 square patio stones (flagstones). You can arrange them in rows to form a variety of rectangular floors for a patio. If s represents the number of stones in a row and n represents the number of rows, what are the possibilities? Fill in a table like this one.

Number of Stones: 240

Number of stones in a row	10	12	15	16	30	40
Number of rows						

(b) Write an equation connecting n, s, and 240. (If you cannot cut any of the stones, what can you say about the kind of numbers n and s must be?)

3. (a) A seesaw will balance if $wd = WD$ when a weight of



w pounds is d feet from the fulcrum and on the other side a weight of W pounds is D feet from the fulcrum. If $WD = 36$, find D when W is 2, 9, or 18 and find W when D is 1, 6, 12.

(b) What happens to W as D is doubled? What happens to D as W increases?

4. Write an equation connecting rate of interest r and the number of dollars on deposit p with a fixed interest payment of \$200 per year. What happens to r when p is doubled? If the interest rate is doubled what can you say of the amount of money on deposit?

5. A length of 20 inches is to be divided into a number of equal parts. Write an equation connecting the number or parts n with the length l of each part. What happens to the

number of parts if the length of each part is divided by 3?

Write the equations in Problems 1 (b), 2 (b), 3, 4, and 5 in a column. Have you noticed the pattern in these examples of inverse variation? Is a constant involved? What is the mathematical sentence which describes inverse variation the way $y = kx$ shows direct variation?

The equation $xy = k$ where k is a constant and not zero states that x varies inversely as y (or that y varies inversely as x).

Definition. If two quantities x and y are connected by the equation $xy = k$ where k is a non-zero constant, then y varies inversely as x (and x varies inversly as y).

Notice that in each case of inverse variation described so far, the constant of proportionality, k , has been a positive number. As in the case of direct variation, k may be negative in special cases but our study is confined to $k > 0$.

Exercises 7-2b

1. State the constant in each of the equations for Problems 1 (b), 2 (b), 3, 4, and 5 of Exercises 7-2a.
2. State your impression of the difference between direct variation and inverse variation.
3. Find k if y varies inversely as x and if y is 6 when x is 2.
4. Find k if x varies inversely as y and if y is 10 when x is $\frac{1}{2}$.

5. From the information in the table does it appear that a varies inversely as b ? Explain your answer.

a	-4	-1	1	3	8	19	41
b	-8	-2	2	6	16	38	82

6. Suppose that a varies inversely as b and that a and b have the corresponding values shown in the table below. Write an equation connecting a and b . Are all the entries in the table needed?

a	4	2	1
b	3	6	12

7. Study the number pairs which follow: (-2, 8); (-1, 2); (0, 0); (1, 2); (2, 8); (3, 18); (4, 32).

   (a) Does it appear that y varies directly as x ?

   (b) Does it appear that y varies inversely as x ?

8. (a) Supply the missing values in the table below where $xy = 18$.

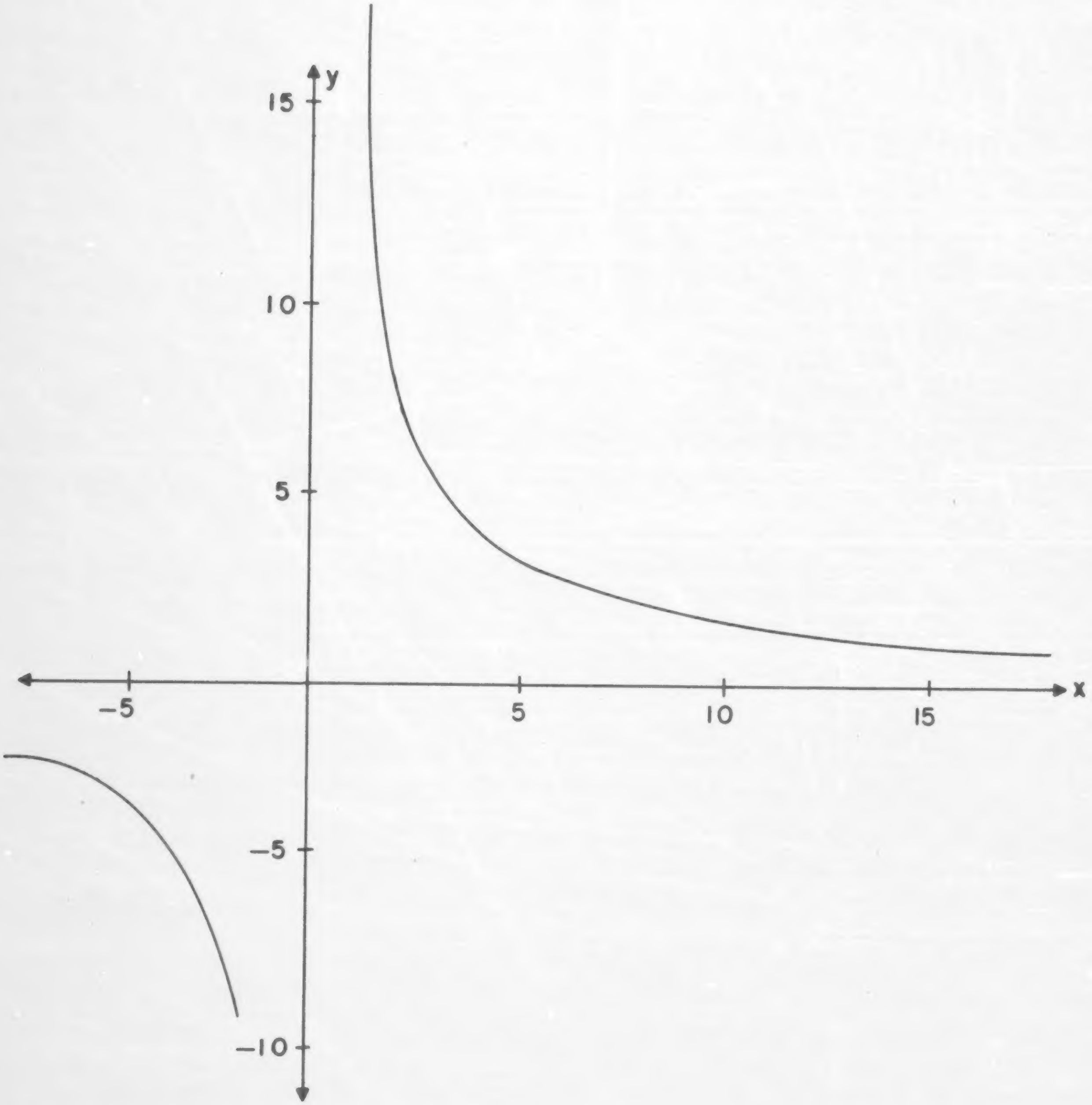
x	-4	-3	-2	-1	1	2	3	4	5	6	7	8	9	18
y														

   (b) Is it possible for x or y to be zero in $xy = 18$? Why?

   (c) Plot on graph paper the points whose coordinates you found in part (a) and draw the curve. You may wish to find more number pairs to enable you to draw the curve more easily. Does your curve look like the one following question 8e?

   (d) When x is doubled is y also doubled? See your graph. What happens to x when y is divided by 6? Are your responses true for negative values of x and y ?

(2) Some of you may have met this curve in the seventh grade in connection with the lever, or balance. The curve, of which your graph is a portion, is called a hyperbola. It has two branches. A hyperbola shows __?__ variation.



9. (a) In the general equation $xy = k$ where k is a constant not equal to zero, may y ever be zero?

   (b) In $xy = K$ what happens to y if x is doubled? What happens to x if y is multiplied by 8? What happens to y if x is divided by 10?

10. (a) On graph paper, draw the graphs of $xy = 1$, $xy = 6$, $xy = \frac{1}{4}$.

   (b) In part (a) what is the intersection set of the three graphs? Did you remember to use negative values for x and y when you plotted points?

   (c) What is the effect of the constant on the graph of $xy = k$?

Notice that in the discussion of direct and inverse variation, the letters x and y may be used interchangeably. In $y = kx$, if k is not zero, we say that x varies as y and y varies as x. In $xy = k$, k cannot be zero and we say that x varies inversely as y or that y varies inversely as x. In these statements x and y are the variables and k represents the constant or fixed value appropriate to the particular case. In the general equation the letter "k" is used rather than a particular numeral, in order to include all possible cases. Since for $x \neq 0$, the equation $xy = k$ may be written $y = k \cdot \frac{1}{x}$ which says that y varies directly as the reciprocal of x.

Occasionally you may see direct variation represented by the statement $y/x = k$. There are times when this form is useful, as in Unit 6, but from your work with zero you know that $y/x = k$ excludes $x = 0$. Hence you may use this form only when you are not interested in the value of zero for x . When $y/x = k$ is used, the number zero cannot be in the domain of the variable x .

The graphs of $y = kx$ and $xy = k$ include points with negative coordinates. In many practical situations these negative numbers do not have a sensible interpretation. The number of stones in Problem 2 of Exercises 7-2a must be a positive whole number. It is not reasonable to speak of a negative number of stones or a negative number of rows. In situations where direction is important, negative numbers help in the interpretation. Negative temperatures are temperatures below zero, for example. Think of a physical interpretation for the equation $d = rt$ which is acceptable for both positive and negative values of t .

7-3. Other Types of Variation

You are familiar with many situations in which one quantity changes when another one does. The cost of filling the gas tank in a car changes when the number of gallons purchased is changed (or when the price of one gallon of gas is changed). Your weight changes when your size changes. The height of a certain tree changes when the number of years it has been growing is changed. The time required to travel a certain distance changes when the speed at which you travel is changed. The total area of a cube changes when the length of a side of the cube is changed. (Of course, all the sides would have to be changed so that the figure remains a cube.) The amount of destruction caused by a certain type of bomb changes when the size of the bomb is changed.

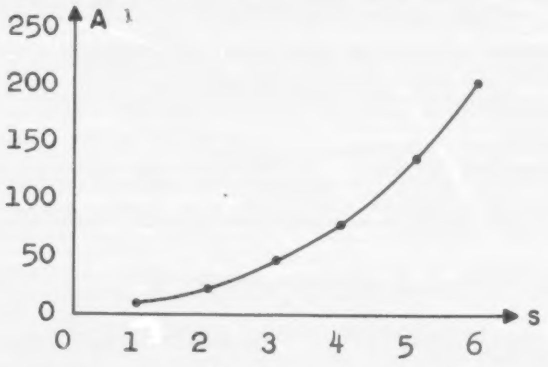
Some of the relationships mentioned above are quite simple, and have been discussed in Sections 7-1 and 7-2. They can be described by one of the equations $y = kx$ or $xy = k$ and we have seen how they can be pictured.

In this section, we shall consider some more complicated relationships and see how they can be described by equations and pictured by graphs. In many applications, negative numbers are not used (for instance, for the area of the surface of a cube), but in some applications a negative value for x or y is meaningful (for instance, for the temperature, or for a person's bank balance). We shall frequently draw the graphs using both positive and negative values of the variables. In later work, you will find that the entire graphs are used in other connections.

Of the relationships mentioned above, many are too difficult for us. In fact, the only new ones which is simple enough for us to consider is the one connecting the area of a cube with the length on one side. For a cube 2 inches on a side, each face is a square. What is the area of one face? How many faces are there? What is the total area of all the faces? It is quite easy to make a table of some of the corresponding values for the total area A sq. in. of the cube and the length s in., of one side. Some of the values are given in the table. Check these values and find the others.

s	A
1	6
2	24
3	
4	
5	
6	216

What information can we get from this short table? Does A vary directly as s ? or inversely as s ? or is this relationship more complicated? Certainly the product $A \cdot s$ is not a constant, so A does not vary inversely as s . Also, A/s is not constant $(6/1 \neq 24/32)$ so A does not vary directly as s . This relationship is more complicated. Let us try to picture it by a graph and describe it by means of an equation, as we have for the simpler relationships of Section 7-1 and 7-2.



The graph is sketched at the right. Since A takes on values from 6 to 216 while the values of s go from 1 to 6, it is convenient to use different scales on the two axes. Graphs appear frequently in newspapers and other publications. Whenever you look at such a graph, be sure to notice what scale is used on each axis. The graph on the previous page would look quite different if the same scale were used on both axes. Sketch a portion of the curve and see. (Use the points (1,6) and (2,24); the other values of A will be too large to be shown on your paper.) Can you find, from the graph on the previous page, the area of a cube 3.5 inches on a side? Can you find the length of one side of a cube whose area is 175 sq. in.? Did you find exact values from the graph?

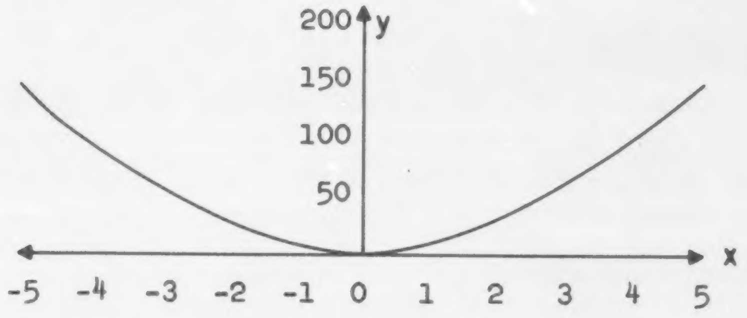
If the length of one side of the cube is s inches, what is the area of one face? How many faces are there? What is the total area of all the faces? You should have found that $A = 6s^2$. Can you find, from the equation, the area of a cube 3.5 inches on a side? Is it easier to use the equation or the graph? Can you think of a case when it would be better to use the graph, and another one when it would be better to use the equation?

Since A and s are connected by the equation $A = 6s^2$, we see that 6 is the constant of proportionality between the corresponding values of A and s^2. Since there is a constant of proportionality between A and s^2, we see that A is proportional to s^2 or A varies directly as s^2.

Definition. If two quantities x and y are related by an equation $y = kx^2$, where k is a constant not zero, then y varies directly as x^2.

This definition is really not new. It is just a restatement, with different symbols, of the definition of direct variation in Section 7-1. The statement "y varies directly as x^2" tells us that $y = kx^2$ for some constant k. It does not tell us the value of K . In some applications, k is a negative number, but we shall consider only positive values of k in this chapter.

The graph of $y = 6x^2$ is drawn below. For positive values of x , we can interpret y as the area in square inches of a cube with side x inches. For negative values of x , the variables cannot be interpreted in this way, but the entire graph is used in more advanced work.



Exercises 7-3a

1. (a) A table in this section shows some corresponding values of the length s inches, or one side of a cube and its area A square inches. You were asked to find the values missing from the table. For each pair of corresponding values in the completed table, find the quotient A/s^2 .

   (b) Use the graph of $y = 6x^2$ drawn in this section to answer the following questions:

      (1) What is the area of a cube of side 2 1/4 inches?

      (2) What is the side of a cube whose area is 75 square inches?

2. Let S be the area in square centimeters of a square with edges e centimeters long.
   (a) Find an equation connecting S and e.
   (b) Tell how S varies with e.
   (c) Plot the graph of the equation you found in part (a). Use values of e from 0 to 15 and choose a convenient scale for the values of S.
   (d) From the graph you drew in part (c), find:
      (1) The area of a square with edges 3 cm. long.
      (2) The length of the edges of a square of area 64 square centimeters.
      (3) The area of a square with edges 5.5 cm. long.
      (4) The length of the edges of a square of area 40 sq. cm.
      (e) From the equation you found in part (a), find:
         (1) The area of a square with edges 3 cm. long.
         (2) The area of a square with edges 5.5 cm. long.
3. If E is proportional to the square of v and E is 64 when v is 4, find:
   (a) An equation connection E and v.
   (b) The value of E when v is 6.
   (c) The value of v when E is 16.
4. Suppose grass seed costs 70 cents per pound, and one pound will sow an area of 280 sq. ft.
   (a) How many pounds of seed will be needed to sow a square plot 10 ft. on a side?
   (b) How much will it cost to buy seed to sow a square plot 10 feet on a side?

(c) If C cents is the cost of the seed to sow a square plot s feet on a side, find an equation connecting C and s.

(d) How much will it cost for seed to sow a square plot 65 feet on a side?

(e) If $15.00 is available for seed, can enough be bought to sow a square plot 75 feet on a side?

5. A ball is dropped from the top of a tower. The distance, d feet, which it has fallen varies as the square of the time t sec., that has passed since it was dropped.

(a) From the information above, what equation can you write connecting d and t ?

(b) Can you find how far the ball falls in the first 3 seconds?

(c) If you are also told that the ball falls 144 feet in the first 3 seconds, what equation can you write connecting d and t ?

(d) Using the equation you wrote in part (c), can you find how far the ball falls in the first 5 seconds?

In the last set of exercises, you found several equations expressing the relationship between two variables. Each of these equations was of the form $y = kx2$, but the constant k had different values in different cases. It sometimes happens that the relationship between two variables is expressed by an equation such as $y = 2x^3$ or $y = 5x^4$, etc. A general equation which includes all of these cases is $y = kx^n$, where k and n are each constants in any one case, but may change from one case to another. In this section, k will be a positive constant and n

will be a counting number. The exercises below illustrate the changes produced in the graph of the equation $y = kx^n$ when either k or n is kept the same and the other is changed. In many situations in science or engineering, two variables are connected by an equation of a different type from $y = kx^n$, but this is the only type of equation discussed in this section.

Exercises 7-3b

1. For each of the equations (a), (b), (c) below, make a table of corresponding values of x and y . Use the values -6, -4, -2, 0, 2x 4, 6 for x .
   (a) $y = \frac{1}{2}x^2$
   (b) $y = x^2$
   (c) $y = 2x^2$
   (d) Complete the following statement: In each of the equations (a), (b), (c) above y varies _____ _____ _____ .

2. Make a horizontal scale for values of x from -6 to 6 and a vertical scale for values of y from 0 to 80. Use this one pair of reference lines to plot all three of the relationships in Problem 1.
   (a) What happens to the graph of $y = kx^2$ if the value of k is doubled?
   (b) What happens to the graph of $y = kx^2$ if the value of k is multiplied by 4?
   (c) What general statements can you make about the change in the graph of $y = kx^2$ caused by changing the value of k ?

3. For each of the three graphs you drew in Problem 2, what happens to y if x is doubled? (Try some values: 1 and 2 ; -2 and -4 ; etc.)

4. For each of the following relationships, make a table of corresponding values of x and y. Use the values -3, -2, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, 2, 2 for x.

   (a) $y = x$ (c) $y = x^3$

   (b) $y = x^2$ (d) $y = x^4$

5. Make a horizontal scale for values of x from -3 to 3 and a vertical scale for values of y from -30 to 50. Use this one pair of reference lines to plot all four of the relationships in Problem 4.

   (a) What general statements can you make about the change in the graph of $y = x^n$ caused by changing the value of n ?

   (b) Compare the graphs of Problem 4 with those of Problem 2. What appears to produce a greater effect on the graph of $y = kx^n$, a change in k , or the same kind of a change in n ?

6. Use the following notation: e cm. is the length of one edge of a cube; P cm. is the perimeter of one face of the cube; S sq. cm. is the total area of all faces of the cube; V cu. cm. is the volume of the cube.

   (a) Find an equation connecting P and e; S and e; V and e.

   (b) Complete the following statements:

      (1) P varies _____ _____ _____.

      (2) S varies _____ _____ _____.

How would you describe the way V varies with e ?

(c) On one set of reference lines, plot the graphs of the the three equations you found in part (a). Use the values 0, 1, 2, 3, 4 for e. (Some of the earlier graphs will help you.)

(d) From the graphs you drew in part (c), find P , S , and V when e is $2\frac{1}{2}$ cm. Check by using the equations you found in part (a).

(e) From the graphs you drew in part (c), can you guess which of P, S , and V will be biggest and which will be smallest when e is 10 cm.? Use the equations you found in part (a) to test your guess.

7-4 Summary and Review

The three kinds of variation considered in this chapter are direct variation, inverse variation, and direct variation as the square.

(1) Direct variation: $y = kx$

(a) If two variables, x and y, are related by the equation $y = kx$, where k is a constant not zero, then y varies directly as x.

(b) The number k is called the constant of proportionality.

(c) When k is positive, as x increases, y must increase and as x decreases, y must decrease.

(d) The graph of $y = kx$ is a straight line through the origin.

(e) The steepness of the line is determined by k.

(2) Inverse variation: $xy = k$

(a) If two variables, x and y, are related by the equation $xy = k$, where k is a constant (not zero) then y varies inversely as x.

(b) The number k is the constant of proportionality between y and the reciprocal of x as shown in the form $y = k \cdot \frac{1}{x}$.

(c) When k is positive, as x increases, y must decrease and as x decreases, y must increase.

(d) The graph of $xy = k$ is not a straight line, but a special curve with two branches. The graph does not go through the origin.

(3) Direct variation as the square: $y = kx^2$

(a) If two variables, x and y, are related by the equation $y = kx^2$, where k is a constant not zero, then y varies directly as x^2.

(b) The number k is called the constant of proportionality between y and x^2.

(c) When k is positive, if x is multiplied by 2, y is multiplied by 4; and if x is multiplied by 3, y is multiplied by 9.

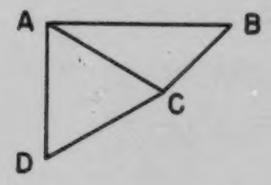
(d) The graph of $y = kx^2$ is not a straight line. The origin is a point on the graph.

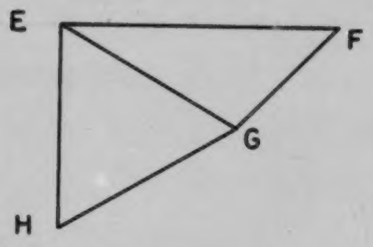
The exercises below review the different types of variation discussed in this chapter.

Exercises 7-4

1. If y varies directly as x, and if y is 16 when x is 2, find y when x is 5.
2. If y varies inversely as x, and if y is 16 when x is 2, find y when x is 5.
3. If y varies directly as the square of x, and if y is 16 when x is 2, find y when x is 5.
4. The areas enclosed by two similar polygons are proportional to the squares of any two corresponding diagonals. The polygons below are similar and point C is 2 centimeters from point A ; point G is 3 centimeters from point E .

Find the ratio of the areas enclosed by the polygons.





5. The distance, d inches, a spring is stretched varies directly as the pull, P pounds, which is applied to the spring.
   (a) If a pull of 10 lbs. stretches a certain spring 5 inches, what pull is required to stretch it 14 inches?
   (b) For the spring in part (a), how far will it be stretched by a pull of 14 lbs.?

6. The pressure, p lbs. per sq. in., exerted by a certain amount of hydrogen gas varies inversely as the volume, v cu. in., of the container in which it is kept. If the pressure is 7 lbs. per sq. in. when the gas is in a gallon jug, what would be the pressure if the gas was enclosed in a half-pint jar?

7. As the altitude increases, the temperature decreases 5.4°F for each 1000 feet of altitude. Is this an example of any of the three types of variation described in this chapter? If your answer is yes, tell which kind and write the equation.

8. Show that, in $x + y = k$, y does not vary inversely as x.

9. The picture editor of a school yearbook often has to change the measurements of a picture in order to place it on a page. Suppose he had a photograph $8\frac{1}{2}$ inches wide and $6\frac{3}{4}$ inches high

which he had to adjust to a space $6\frac{1}{2}$ inches wide by having it reduced in size. What height should the altered picture have? This is an example of what kind of variation?

10. In $y = k + x$, y increases as x increases. Show that this is not an example of direct variation.

11. If a photograph $8\frac{1}{2}$ inches wide by 11 inches long is to be enlarged to 15 inches in length, what must the new width be?

12. If A is 24 in $A = lw$, what kind of variation is indicated between l and w?

13. What kind of variation is represented by $C = \pi d$? What is the constant of proportionality?

14. In $V = \pi r^2 h$, suppose r is multiplied by 5 while h is unchanged. What happens to V? What is the constant in this case?

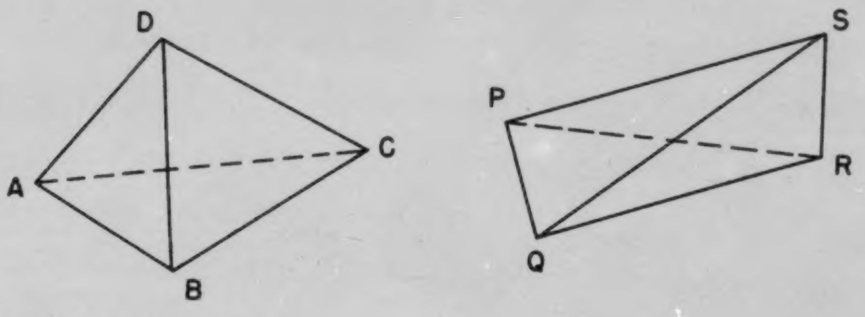
15. The cost in cents of n two-cent stamps is given by the equation $c = 2n$. Find 5 number pairs (n, c) which satisfy this equation. What would the graph of this relation look like?

UNIT 8

NON-METRIC POLYHEDRONS

8-1. Tetrahedrons

A geometric figure of a certain type is called a tetrahedron. A tetrahedron has four vertices which are points in space. The drawings below represent tetrahedrons. (Another form of the word "tetrahedrons" is "tetrahedra".

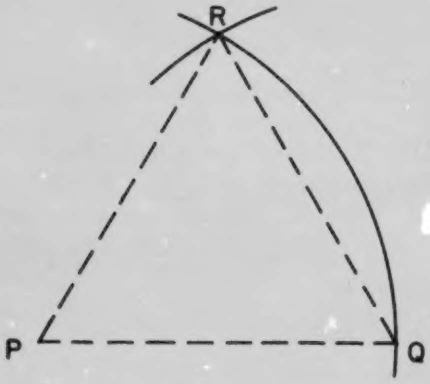


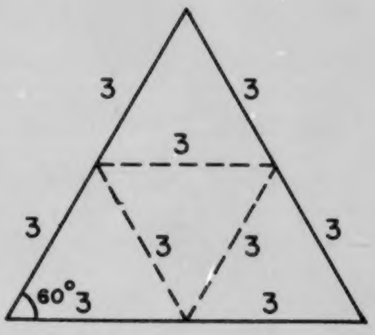
The points A, B, C, and D are the vertices of the tetrahedron on the left. The points P, Q, R, and S are the vertices of the one on the right. The four vertices of a tetrahedron are not in the same plane. The word "tetrahedron" refers either to the surface of the figure or to the "solid" figure, i. e. the figure including the interim in space. In this first section the distinction is not important. Later we shall use the term "solid tetrahedron" when we mean the surface together with the interior. We can name a tetrahedron by naming its vertices. We shall normally put parentheses around the letters like (ABCD) or (PQRS). Later we shall use this notation to mean "solid tetrahedron".

The segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ are called the edges of the tetrahedron (ABCD). We sometimes will use the notation (AB) to mean the edge $\overline{AB}$. What are the edges of the tetrahedron (PQRS)?

Any three vertices of a tetrahedron are the vertices of a triangle and lie in a plane. A triangle has an interior in the plane in which its vertices lie (and in which it lies). Let us use (ABC) to mean the triangle ABC together with its interior. In other words, (ABC) is the union of Δ ABC and its interior. The sets (ABC), (ABD), (ACD), and (BCD) are called the faces of the tetrahedron (ABCD). What are the faces of the tetrahedron (PQRS)?

You will be asked to make some models of tetrahedrons in the exercises. The easiest type of tetrahedron of which to make a model is the so-called regular tetrahedron. Its edges are all the same length. (We introduce length or measurement here only for convenience in making some uniform models. This chapter, like Chapter 4 of Volume I, deals fundamentally with non-metric or "no-measurement" geometry.) On a piece of cardboard or stiff paper construct an equilateral triangle of side 6". You can do this with a ruler and compass or with a ruler and protractor.

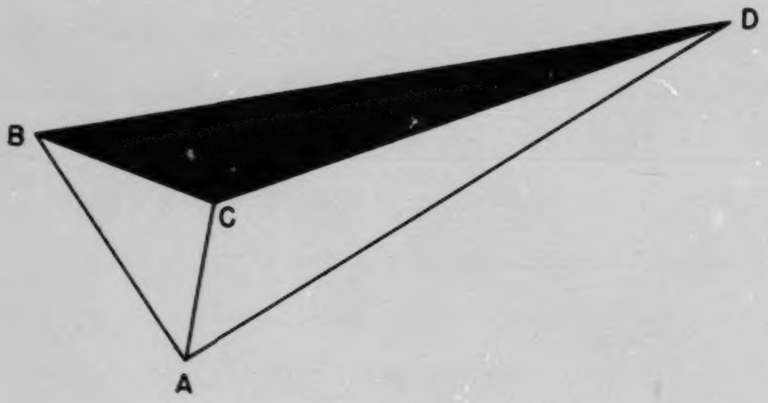




Can you see how the drawing on the left above suggests the construction with ruler and compass? The arc through Q and R has center at P. The other arc through R has the same radius, and center at Q. The segments $\overline{PQ}$, $\overline{PR}$, and $\overline{QR}$ have the same lengths.

Now mark the three points that are halfway between the various pairs of vertices. Cut out the large triangular region carefully. Carefully make three folds or creases along the segments joining the "halfway" points. See the figure on the right above. Use a ruler or other straightedge to help you make these folds. Your original triangular region now looks like four smaller triangular regions. Bring the original three vertices together above the center of the middle triangle. Fasten the loose edges together with tape or paper and paste. You now have a model of a regular tetrahedron.

How do we make a model of a tetrahedron which is not a regular one? We can do it something like this. Cut any triangular region out of cardboard or heavy paper. Use this as the base of your model. Label its vertices A, B, and C. Cut out another triangle with one of its edges the same length as $\overline{AB}$. Now, with tape, fasten these two triangles together along edges of equal length. Use edge (AB) for this, for instance. Two of the vertices of the second triangle are now considered labeled A and B. Label the other vertex of the second triangle D. Cut out a third triangular region with one edge the length of $\overline{AD}$ and another the length of $\overline{AC}$. Do not make the angle between these edges too large or too small. Now, with tape, fasten these edges of the third triangle to $\overline{AD}$ and $\overline{AC}$ so that the three triangles fit together in space. The model you have constructed so far will look something like a conical drinking cup if you hold the vertex A at the bottom. Finally cut out a triangular region which will just fit the top, fasten it to the top and you will have your tetrahedron.



Exercises 8-1

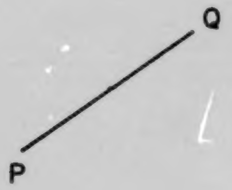
1. Make 2 separate cardboard or heavy paper models of a regular tetrahedron. Make your models so that their edges are each 3" long. Follow the instructions in the text.
2. Make two models of tetrahedrons which are not regular.

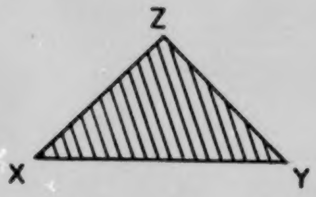
*3. In making the third face of a non-regular tetrahedron, what difficulties would we encounter if we made the angle DAC too large or too small?

8-2. Simplexes

What is the simplest object or set of points you can think of? How about a single point? Most of you would agree that you couldn't find anything much simpler than that. What would be the next most simple set of points in space? Some of you probably would say "a set of two points" and that is a good suggestion. But any two different points in space are on exactly one line. Also any two different points in space are the endpoints of exactly one segment. Thus, the set of two points determines two other simple sets in space: a line and a segment. The segment is a part of the line which contains the points. A segment has length but does not have area. We speak of a segment as being one-dimensional. It can be considered as the simplest one-dimensional object in space. (A line also might be, but in this chapter we want to think about the segment, not the line.)

What is the next most simple set of points in space? The answer we expect is probably clear now, and you say "a set of three points". What do three points in space determine? If the three points are all on the same line, then we get just a part of a line. We are not much better off than we were with just two points. Let us agree, therefore, that our three points are not to be on the same line. Then there is exactly one plane containing the three points and there is exactly one triangle with the three points as vertices. There is also exactly one triangular region which together with the triangle which bounds it, has the three points as vertices. This mathematical object, the triangle, together with its interior, is what we want to think about. It has area and it is two-dimensional. It can be considered as the simplest two-dimensional object in space.





It seems rather clear that the next most simple set of points in space would be a set of four points. If the four points were all in one place then the figure determined by the four points would apparently also be in one plane. We want to require that the four points are not all in any one plane. This requirement also guarantees us that no three can be on a line. If any three were on a line then there would be a plane containing that line and the fourth point and the four points would be in the same plane. We have four points in space, then, not all in the same plane. What kind of object does this suggest? You are right if you thought of the tetrahedron. The four points in space are the vertices of exactly one (solid) tetrahedron. A solid tetrahedron has volume and it is three-dimensional. It can be considered as the simplest three-dimensional object in space.

Here we have four objects each of which may be thought of as the simplest of its kind. There are remarkable similarities among these objects. They all ought to have names that sound alike and remind us of their basic properties. We call each of these a simplex. How do we tell them apart? We label each with its natural dimension. Thus a set consisting of a single point is called a 0-simplex. A segment is called a 1-simplex. A triangle together with its interior is called a 2-simplex. A solid tetrahedron is called a 3-simplex.

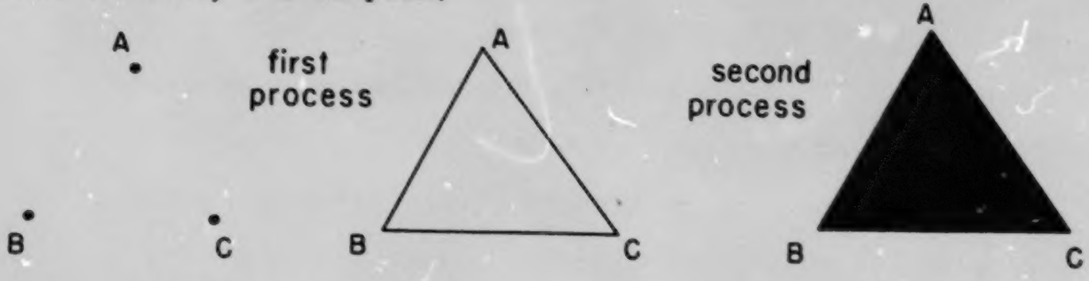
Let us make up a table to help us keep these ideas in order.

A set consisting of:	determines:	It is called a:
one point	one point (itself)	0-simplex
two points	a segment	1-simplex
three points not all on any one line	a triangle together with its interior	2-simplex
four points not all on any one plane	a solid tetrahedron (which includes its interior)	3-simplex

There is another way to think about the dimension of these sets. In this we think of the notion of betweenness, of a point being between two other points.

Let us start with two points. Consider these two points and all points between them. We now have a segment. Now take the segment together with all points which are between any two points of the segment. We still have the same segment. No new points were obtained by "taking points between" again. The process of "taking points between" needed to be used just once. We get a one-dimensional set, a 1-simplex.

Next consider three points not all on the same line. Then let us apply our process. We take these points together with all points which are between any two of them. At this stage we have a triangle but not its interior. We apply the process again. We take the set we already have (the triangle) together with all points which are between any two points of this set. We get the union of the triangle and its interior. If we apply the process again we don't get anything new. We need use the process just twice. We get a two-dimensional set, a 2-simplex.



Next let us consider four points not all on the same plane. We apply the process of "taking points between" and get the union of the edges of a tetrahedron. We apply the process again and get the union of the faces. We apply it once more and get the solid tetrahedron itself. We apply it again and still get just the solid tetrahedron. We need use the process just three times. We get a three-dimensional set, a 3-simplex.

If we had just one point, the application of the process would still leave us with just the one point. We need apply the process zero times. We get a zero-dimensional set, a 0-simplex. (We mention this case last because we have to understand the process before it can make much sense.)

Finally, let us consider a 3-simplex. Look at one of your models of tetrahedrons. It has four faces and each face is a 2-simplex. How nice! It has six edges and each edge is a 1-simplex. It has four vertices and each vertex is a 0-simplex.

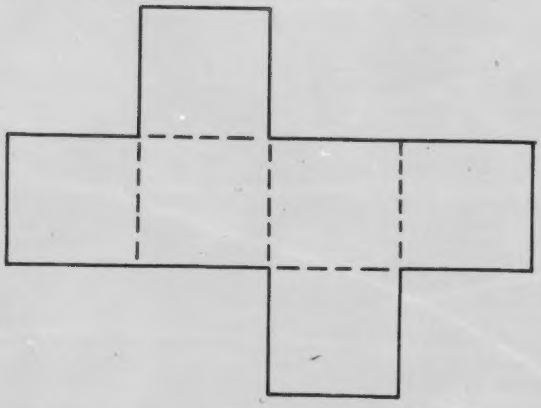
Exercises 8-2

1. (a) A 2-simplex has how many 1-simplexes as edges?
   (b) It has how many 0-simplexes as vertices?
2. A 1-simplex has how many 0-simplexes as vertices?
3. Using models show how two 3-simplexes can have an intersection which is exactly a vertex of each.
4. Using models show how two 3-simplexes can have an intersection which is exactly an edge of each.
5. In this and the next problem you are asked to do a bit of coloring. Mark three points not all on the same line in blue. Color red all points which are between any two of these. Shade green all points which are between any two of the points already colored. Should there be any points which are not colored and are between two of the colored points? Starting with the three points, how many times did you need to use the process of "taking points between" before you were finished?
6. Use one of your models of a non-regular tetrahedron.

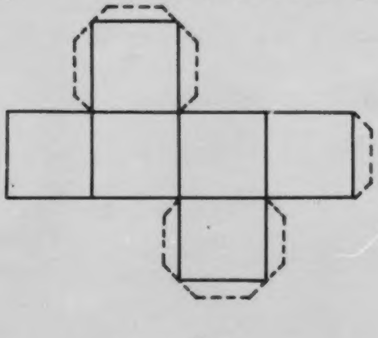
Color its vertices blue. Color red the set of all points each of which is between two of the vertices. Color green the set of all points each of which is between two of the red or blue colored points. You should now have your model colored. What is the set of all points which either are colored or are between two of your colored points?

8-3. Models of Cubes

Most of you already know that if you want to make an ordinary box you need six rectangular faces for it. They have to fit and you have to put them together right. There is a rather easy way to make a model of a cube.



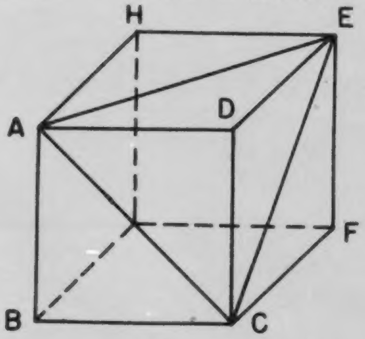
Draw six squares on heavy paper or cardboard as in the drawing above. Cut around the boundary of your figure and fold (or crease) along the dotted lines. Use cellulose tape or paste to fasten it together. If you are going to use paste it will be necessary to have flaps as indicated in the drawing below.

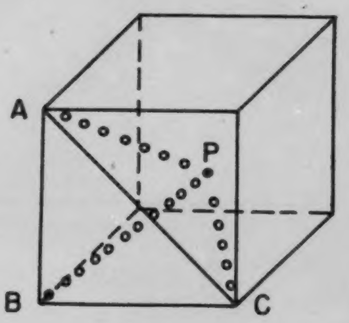


You will be asked to make two models of a cube in the exercises.

Can the surface of a cube be regarded as the union of 2-simplexes (that is, of triangles together with their interiors)? Can a solid cube be regarded as the union of 3-simplexes (that is of solid tetrahedrons)? The answer to both of these questions is "yes". We shall explain one way of thinking about these questions.

Each face of a cube can be considered to be the union of two 2-simplexes. The drawing on the left below shows a cube with three of its faces subdivided into two 2-simplexes each. The face ABCD appears as the union of (ABC) and (ACD) for example. The other faces which are indicated as subdivided are CDEF and ADEH. We can think of each of the other faces as the union of two 2-simplexes. Thus the surface of the cube can be thought of as the union of twelve 2-simplexes.





With the surface regarded as the union of 2-simplexes we may regard the solid cube as the union of 3-simplexes (solid tetrahedrons) as follows. Let P be any point in the interior of the cube. For any 2-simplex on the surface, (ABC), for example, (PABC) is a 3-simplex. In the figure on the right above, P is indicated as inside the cube. The 1-simplexes (PA), (PB), and

(PC) would also be inside the cube. Thus with twelve 2-simplexes on the surface, we would have twelve 3-simplexes whose union would be the cube. The solid cube is the union of 3-simplexes in this nice way.

Now we ask another question. Do you suppose that a 3-simplex can be regarded as the union of a certain number of solid cubes? Can we find solid cubes that will fit together to fill up a 3-simplex? The answer to these questions is no. Suppose cubes could be fitted together to fill up a 3-simplex. Then any face of the 3-simplex would be filled up by square regions which are faces of the cubes. The square regions have right angles at their vertices. Any face of a 3-simplex is triangular. At least two of the angles of a triangle must be less than a right angle. Therefore the square regions cannot fit. A 3-simplex cannot be a finite union of cubes.

Exercises 8-3

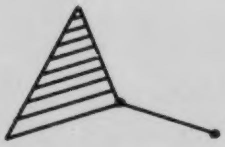
1. Make two models of cubes out of cardboard or heavy paper. Make them with each edge 2" long.
2. On one of your models, without adding any other vertices, draw segments to express the surface of the cube as a union of 2-simplexes. Label all the vertices on the model A, B, C, D, E, F, G, and H. Think of a point P in the interior of the cube. Using this point and the vertices of the 2-simplexes on the surface list the eight 3-simplexes whose union is the solid cube.
3. On the same cube as in problem 2, mark a point in the center of each face. (Each should be on one of the segments you drew

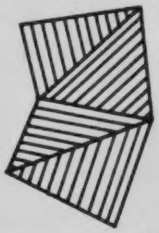
in problem 2.) Draw segments to indicate the surface of the cube as the union of 2-simplexes using as vertices the vertices of the cube and these six new points you have marked. The surface is now expressed as the union of how many 2-simplexes?

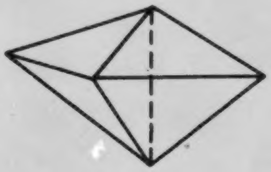
4. Think about a polyhedron formed by putting a square-based pyramid on each face of a cube. The surface of this new polyhedron has how many triangular faces? Can you compare this new polyhedron vertex for vertex, edge for edge, and 2-simplex for 2-simplex with the surface of the cube subdivided into 2-simplexes as in problem 3?

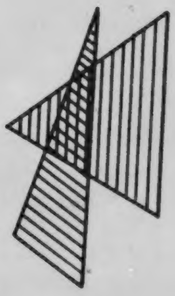
8-4. Polyhedrons

A polyhedron is the union of a finite number of simplexes. It is just one simplex, or maybe the union of seven simplexes, or maybe of 7,000,000 simplexes. What we are saying is that it is the union of some certain number of simplexes. In the previous section, we observed that a solid cube, for example, was the union of twelve 3-simplexes. The figures below represent the unions of simplexes.





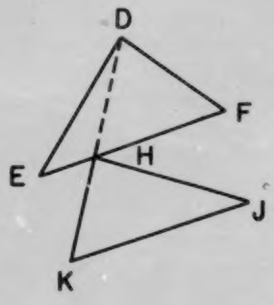


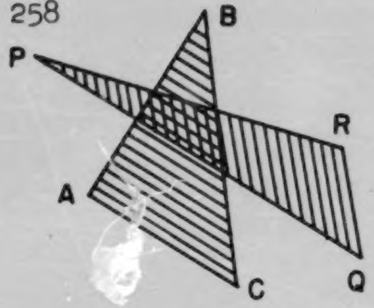


The figure on the left represents a union of a 1-simplex and a 2-simplex which does not contain the 1-simplex. It is therefore of mixed dimension. In what follows, we shall not be concerned with polyhedrons (or polyhedra) of mixed dimension. We assume a polyhedron is the union of simplexes of the same dimension. We shall speak of a 3-dimensional polyhedron as one which is the union of 3-simplexes. A 2-dimensional polyhedron is one which is the union of 2-simplexes. A 1-dimensional polyhedron is one which is the union of 1-simplexes. (Any finite set of points could be thought of as a 0-dimensional polyhedron but we won't be dealing with such here.)

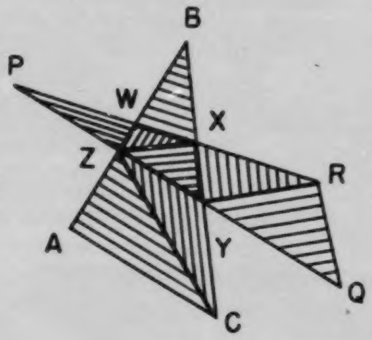
The figure on the right above represents a polyhedron which seems to be the union of two 2-simplexes (triangular regions) but they don't intersect nicely. We prefer to think of a polyhedron as the union of simplexes which intersect nicely as in the middle two figures. Just what do we mean by simplexes intersecting nicely? There is an easy explanation for it. If two simplexes of the same dimension intersect nicely, then the intersection must be a face, or an edge, or a vertex of each.

Let us look more closely at the union of simplexes which do not intersect nicely. In the figure on the right the 2-simplexes (DEF) and (HJK) have just the point H in common. They do not intersect nicely. While H is a vertex of (HJK), it is not of (DEF). However, the polyhedron which is the union of these two 2-simplexes is also the union of three 2-simplexes which do intersect nicely, namely, (DEH), (DHF), and (HJK).





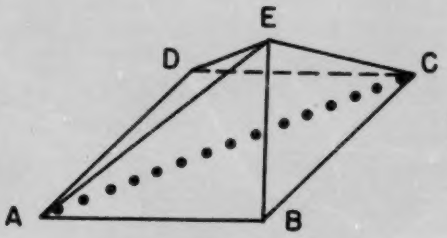
The figure on the left represents the union of the 2-simplexes (ABC) and (PQR). They do not intersect nicely. Their intersection seems to be a quadrilateral together with its interior. On the right we have indicated how the same set of points (the same polyhedron) can be considered to be a finite union of 2-simplexes which do intersect nicely. The polyhedron is the union of the eight 2-simplexes, (ACZ), (CZY), (PZW), (XYZ), (WXZ), (BWX), (XYR), and (YQR).

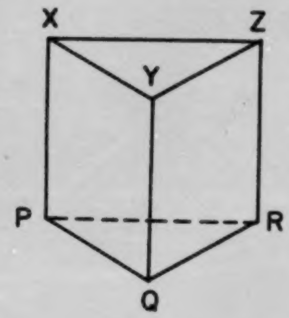


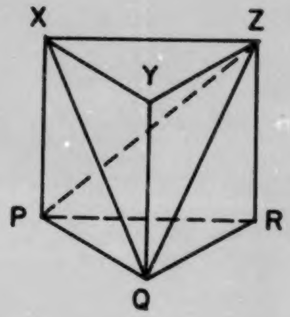
These examples suggest a fact about polyhedrons. If a polyhedron is the union of simplexes which intersect any way at all then the same set of points (the same polyhedron) is also the union of simplexes which intersect nicely. Except for the exercises at the end of this section, we shall always deal with unions of simplexes which intersect nicely. We will regard a polyhedron as having associated with it a particular set of simplexes which intersect nicely and whose union it is. When we say the word "polyhedron", we understand the simplexes to be there.

Is a solid cube a polyhedron, that is, is it a union of 3-simplexes? We have already seen that it is. Is a solid prism a polyhedron? Is a solid square-based pyramid? The answer to all of these questions is yes. In fact, any solid object each of whose faces is flat (that is, whose surface does not contain any curved portion) is a 3-dimensional polyhedron. It can be expressed as the union of 3-simplexes.

As examples let us look at a solid pyramid and a prism with a triangular base.







In the figure on the left the solid pyramid is the union of the two 3-simplexes (ABCE) and (ACDE). The figure in the middle represents a solid prism with a triangular base. The prism has three rectangular faces. Its bases are (PQR) and (XYZ). In Unit 9 you will see a way of expressing a solid prism like this as the union of three 3-simplexes. Here we see how it may be expressed as the union of eight 3-simplexes.

We use the same device we used for the solid cube. First we think about the surface as the union of 2-simplexes. We already have the bases as 2-simplexes. Then we think of each rectangular face as the union of two 2-simplexes. In the figure on the right above the face YZRQ is indicated as the union of (YZQ) and (QRZ), for instance. Now think about a point F in the interior of the prism. The 3-simplex (FQRZ) is one of eight 3-simplexes each with F as a vertex and whose union is the solid prism. In the exercises you will be asked to name the other seven.

Finally, how do we express a solid prism with a non-triangular base as a 3-dimensional polyhedron (that is, as a union of

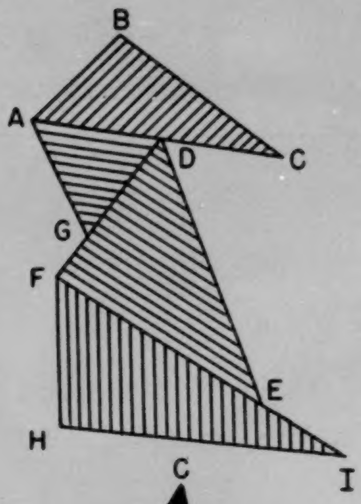
3-simplexes with nice intersections)? We use a little trick. We first express the base as a union of 2-simplexes and therefore the solid prism as a union of triangular solid prisms. And we can then express each triangular solid prism as the union of eight 3-simplexes. We can do this in such a way that all the simplexes intersect nicely.

There is a moral to our story here. To do a harder-looking problem, we first try to break it up into a lot of easy problems each of which we already know how to do (or at least are able to do).

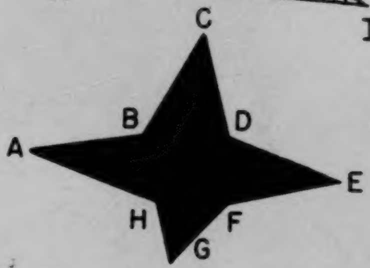
Exercises 8-4

1. Draw two 2-simplexes whose intersection is one point and
   (a) is a vertex of each .
   (b) is a vertex of one but not of the other.
2. Draw three 2-simplexes which intersect nicely and whose union is itself a 2-simplex. (Hint: start with a 2-simplex as the union and subdivide it.)
3. You are asked to draw various 2-dimensional polyhedrons each as the union of six 2-simplexes. Draw one such that
   (a) No two of the 2-simplexes intersect.
   (b) There is one point common to all the 2-simplexes but no other point is common to any pair.
   (c) The polyhedron is a square together with its interior.

4. The figure on the right represents a polyhedron as the union of 2-simplexes without nice intersections. Draw a similar figure yourself and then draw in three segments to make the polyhedron the union of 2-simplexes which intersect nicely.



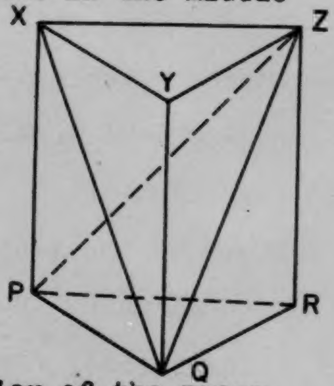
5. The 2-dimensional figure on the right can be expressed as a union of simplexes with nice intersections in many ways. Draw a similar figure yourself.



(a) By drawing segments express it as the union of six 2-simplexes without using more vertices.

(b) By adding one vertex near the middle (in another drawing of the figure), express the polyhedron as the union of eight 2-simplexes all having the point in the middle as one vertex.

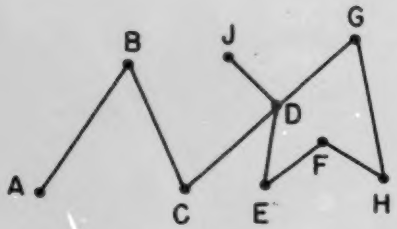
6. (a) List eight 2-simplexes whose union is the surface of the triangular prism on the right. (The figure is like that used earlier.)

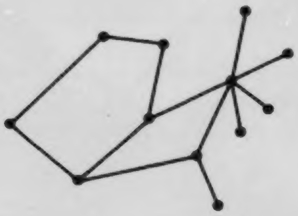


(b) Regarding F as a point in the interior of the prism list eight 3-simplexes (each containing P) whose union is the solid prism.

8-5. One-Dimensional Polyhedrons

A 1-dimensional polyhedron is the union of a certain number of 1-simplexes (segments). A 1-dimensional polyhedron may be contained in a plane or it may not be. Look at a model of a tetrahedron. The union of the edges is a 1-dimensional polyhedron. It is the union of six 1-simplexes. It does not lie in a plane. We may think of the figures below as representing 1-dimensional polyhedrons that do lie in a plane (the plane of the page).

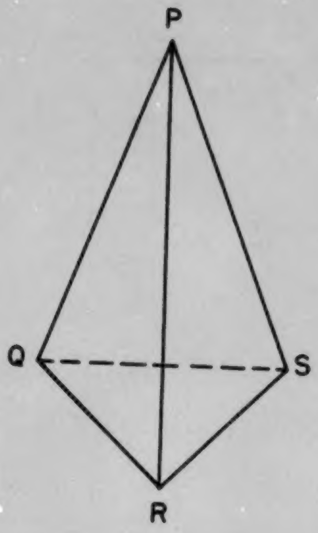




There are two types of 1-dimensional polyhedrons which are of special interest. A *polygonal path* is a 1-dimensional polyhedron in which the 1-simplexes can be considered to be arranged in order as follows. There is a first one and there is a last one. Each other 1-simplex of the polygonal path has one vertex in common with the 1-simplex which precedes it and one vertex in common with the 1-simplex which follows it. There are no extra intersections. The first and last vertices (points) of the polygonal path are called the endpoints.

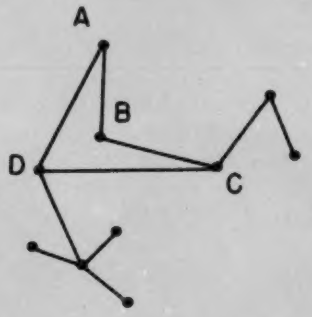
Neither of the 1-dimensional polyhedrons in the figures above is a polygonal path. But each contains many polygonal paths. The union of (AB), (BC), (CD), (DG) and (GH) is a polygonal path from A to H. The union of (JD) and (DE) is a polygonal path from J to E and consists of just two 1-simplexes.

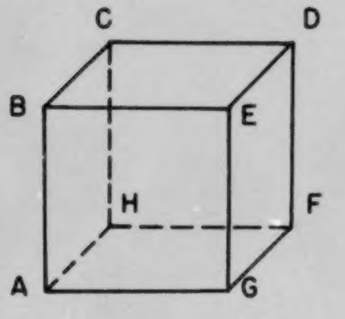
In the drawing on the right of a tetrahedron, the union of (PQ), (QR), and (RS) is a polygonal path from P to S (with endpoints P and S). The 1-simplex (PS) is itself a polygonal path from P to S. Consider the 1-dimensional polyhedron which is the union of the edges of the tetrahedron. Find three other polygonal paths from P to S in it. (Use a model if it hels you see it.)



The union of two polygonal paths that have exactly their endpoints in common is called a simple closed polygon (it is also a simple closed curve). Another way of describing a simple closed polygon is to say that it is a 1-dimensional polyhedron which is in one piece and has the property that every vertex of it is in exactly two 1-simplexes of it.

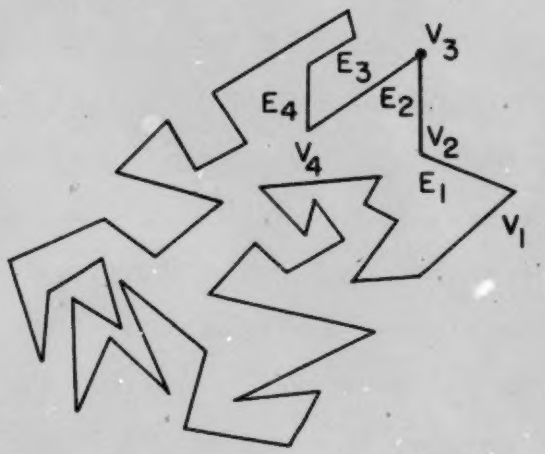
The 1-dimensional polyhedron on the right is not a simple closed polygon. But it contains exactly one simple closed polygon, namely the union of (AB), (BC), (CD), and (DA). Note that (DA) is (AD).





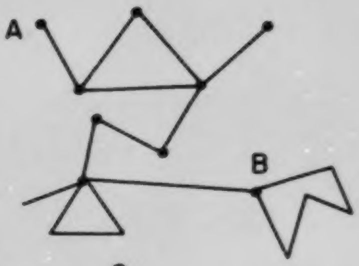
The union of the edges of the cube in the drawing on the left is a 1-dimensional polyhedron. It contains many simple closed polygons. One is the union of (AB), (BE), (EG), and (GA). Another is the union of (AB), (BC), (CD), (DE), (EG), and (GA). Can you give at least two more simple closed polygons containing (BE) and (GA)? (Use a model if it helps you see it.)

There is one very easy relationship on any simple closed polygon. The number of 1-simplexes (edges) is equal to the number of vertices. Consider the figure on the right. Suppose we start at some vertex. Then we take an edge containing this vertex. Next we take the other vertex contained in this edge and then the other edge containing this second vertex. We may think of numbering the vertices and edges as in the figure. We continue the process. We finish with the other edge which contains our original vertex. We start with a vertex and finish with an edge after having alternated vertices and edges as we go along. Thus the number of vertices is the same as the number of edges.



Exercises 8-5

1. The figure on the right represents a 1-dimensional polyhedron. How many polygonal paths does it contain with endpoints A and B? How many simple closed polygons does it contain?

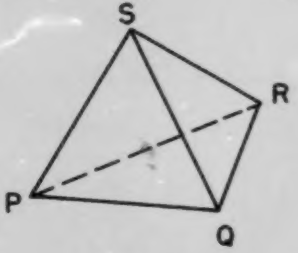


2. (a) The union of the edges of a 3-simplex (solid tetrahedron) contains how many simple closed polygons?

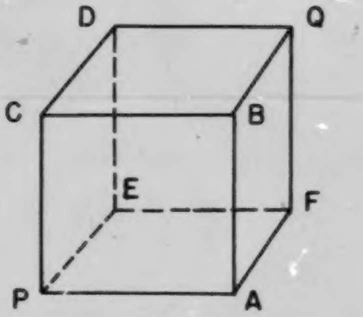
   (b) Name them all.

   (c) Name one that is not contained in a plane.

   (Use a model if you wish.)



3. Let P and Q be vertices of a cube which are diametrically opposite each other (lower front left and upper back right). Name three polygonal paths from P to Q each of which contains all the vertices of the cube and is in the union of the edges. (Use a model if you wish.)



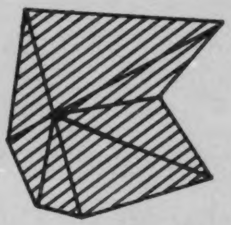
4. Draw a 1-dimensional polyhedron which is the union of seven 1-simplexes and contains no polygonal path consisting of more than two of these simplexes.

5. Draw a simple closed polygon on the surface of one of your

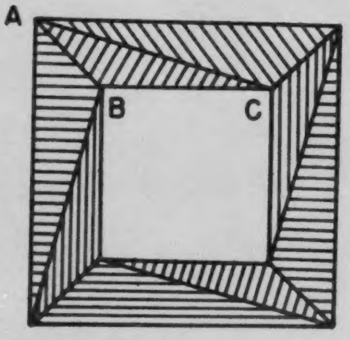
models of a cube which intersects every face and which does not contain any of the vertices of the cube.

8-6. Two-Dimensional Polyhedrons

A 2-dimensional polyhedron is a union of 2-simplexes. As in section 8-4, we will agree that the 2-simplexes are to intersect nicely. That is, if two 2-simplexes intersect, then the intersection is either an edge of both, or a vertex of both. There are many 2-dimensional polyhedrons. Some are in one plane but many are not in any one plane. The surface of a tetrahedron, for instance, is not in any one plane. Let us first consider a few 2-dimensional polyhedrons in a plane. In drawing 2-simplexes in a plane we shall shade their interiors.

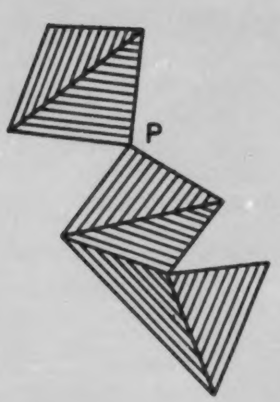


Every 2-dimensional polyhedron in a plane has a boundary in that plane. The boundary is itself a 1-dimensional polyhedron. The boundary may be a simple closed polygon as in the figure on the right. In the figure on the left below we have indicated a polyhedron as the union of eight 2-simplexes. (ABC) is one of them.



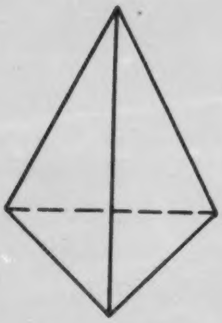
The boundary is the union of two simple closed polygons, the inner square and the outer square. The two polygons do not intersect.

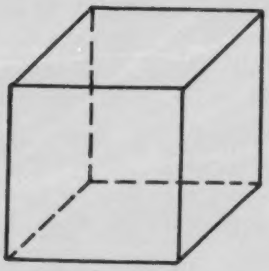
The figure on the right represents a 2-dimensional polyhedron which is the union of six 2-simplexes. The boundary of this polyhedron in the plane is the union of two simple closed polygons which have exactly one vertex of each in common, the point P.

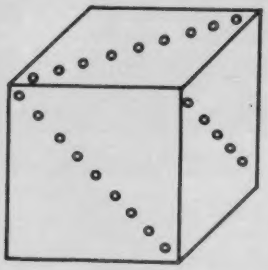


Suppose a 2-dimensional polyhedron in the plane has a boundary which is a simple closed polygon (and nothing else). Then the number of 1-simplexes (edges) of the boundary is equal to the number of 0-simplexes (vertices) of the boundary. You have already seen, in the previous section, why this must be true.

There are many 2-dimensional polyhedrons which are not in any one plane. The surface of a tetrahedron is such a polyhedron. The surface of a cube is another (it may be considered to be expressed as a union of 2-simplexes). Here we have some 2-dimensional polyhedrons which are themselves surfaces or boundaries of 3-dimensional polyhedrons. Let us consider these two surfaces, the surface of a tetrahedron and the surface of a cube.







You may look at the drawings above or you may look at some models (or both). Let us count the number of vertices, the number of edges and the number of faces. But the surface of a cube can be considered in at least two different ways. We can think of the faces as being square regions (as in the middle figure) or we may think of each square face as subdivided into two 2-simplexes (as in the figure on the right). We will use F for the number of faces, E for the number of edges and V for the number of vertices. If you are counting from models and do not observe patterns to help you count, it is usually easier to check things off as you go along. That is, mark the objects as you count them.

Let us make up a table of our results.

	F	E	V
Surface of tetrahedron	?	6	?
Surface of cube (square faces)	?	?	8
Surface of cube (two 2-simplexes on each square face)	12	?	?

It is not easy from just these three examples to observe any nice relationship among these numbers. What we are looking for is a relationship which will be true not only for these 2-dimensional polyhedrons but also for others like these. Try and see if you can guess the relationship we will be telling you about in section 8-8.

Exercises 8-6

1. Make up a table as in the text showing F, V, and E for the 2-dimensional polyhedrons mentioned there.
2. Draw a 2-dimensional polyhedron in the plane with the

polyhedron the union of ten 2-simplexes such that

(a) its boundary is a simple closed polygon,

(b) its boundary is the union of three simple closed polygons having exactly one point in common,

(c) its boundary is the union of two simple closed polygons which do not intersect.

3. Draw a 2-dimensional polyhedron in the plane with the number of edges in the boundary

   (a) equal to the number of vertices,

   (b) one more than the number of vertices,

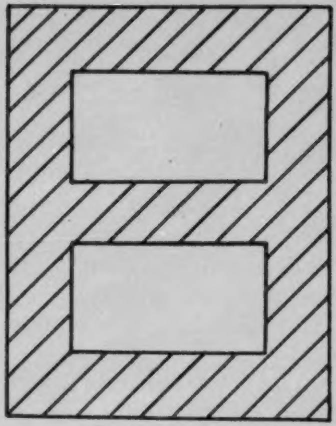
   (c) two more than the number of vertices.

4. Draw a 2-dimensional polyhedron which is the union of three 2-simplexes with each pair having exactly an edge in common. Do you think that there exist in the plane a polyhedron which is the union of four 2-simplexes such that each pair have exactly an edge in common?

5. On one of your models of a cube, mark six points one at the center of each face. Consider each face to be subdivided into four 2-simplexes each having the center point as a vertex. Count F (the number of 2-simplexes), E (the number of 1-simplexes), and V (the number of 0-simplexes) for this subdivision of the whole surface. Keep your answers for later use.

*6. Do the problem above without using a model and without doing any actual counting. Just figure out how many of each there must be. For instance, there must be 14 vertices, 8 original ones and 6 added ones.

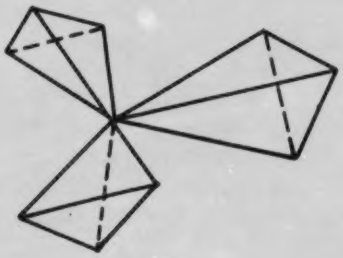
*7. Express the polyhedron on the right as a union of 2-simplexes which intersect nicely (in edges or vertices of each other).



8-7. Three-Dimensional Polyhedrons

A 3-simplex is one 3-dimensional polyhedron. A solid cube is another 3-dimensional polyhedron. Any union of a certain number of 3-simplexes is a 3-dimensional polyhedron. We will assume again that the simplexes of a polyhedron intersect nicely. That is, that if two 3-simplexes intersect, the intersection is a 2-dimensional face (2-simplex) of each or an edge (1-simplex) of each or a vertex (0-simplex) of each.

Any 3-dimensional polyhedron has a surface (or boundary) in space. This surface is itself a 2-dimensional polyhedron. It is the union of several 2-simplexes (which intersect nicely). The surface of the 3-dimensional polyhedron represented by the drawing on the right is something of a mess. It consists of the surfaces of three tetrahedrons which have exactly one point in common.



The simplest kinds of surfaces of 3-dimensional polyhedrons are called simple surfaces. The surface of a cube and the surface of a 3-simplex are both simple surfaces. There are many others. Any surface of a 3-dimensional polyhedron obtained as follows will be a simple surface. Start with a solid tetrahedron. Then fasten another to it so that the intersection of the one you are adding with what you already have is a face of the one you are adding. You may keep adding more solid tetrahedrons in any combination or of any size provided that each one you add in turn intersects what you already have in a set which is exactly a union of one, two, or three faces of the 3-simplex you are adding. The surface of any polyhedron formed in this way will be a simple surface.

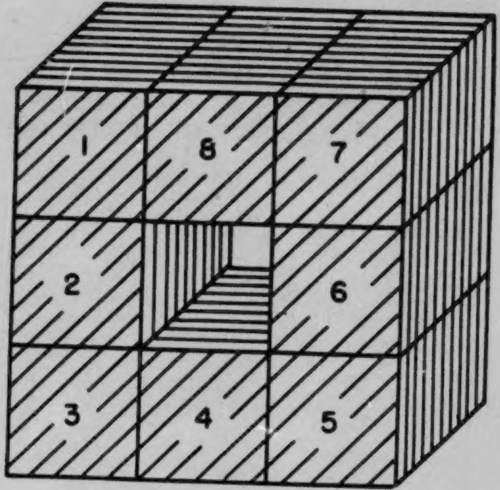
Class activity. Take five of the models of the regular tetrahedrons of edges 3" that the members of the class have made. Put marks on all four faces of one of these. Now fasten each of the others in turn to one of the marked faces. The marked one should be in the middle and you won't see it any more. The surface of the object you have represents a simple surface. You can see how to fasten a few more tetrahedrons on to get more and more peculiar looking objects. Suppose it is true that whenever you add a solid tetrahedron the intersection of what you add with what you already have is one face, two faces or three faces of the one you add. The surface of what you get will be a simple surface.

One can also fasten solid cubes together to get various 3-dimensional polyhedrons. In fastening solid cubes in turn onto what you already have, you will always wind up with a 3-dimensional polyhedron which has a simple surface provided the following condition is met. The intersection of each one you add in turn with what you already have must be a set which is bounded on the surface of the cube you are adding by a simple closed polygon. For example, the intersection might be a face or the union of two adjacent faces of the one you add.

Finally we mention an interesting property of simple surfaces. Draw any simple closed polygon on a simple surface. Then this polygon separates the simple surface into exactly two connected pieces.

Class activity. On the surface of one of the peculiar 3-dimensional polyhedrons (with simple surface) that you have constructed above, have a student draw any simple closed polygon. The wilder the better. It need not be in just one face. Then have another student start coloring somewhere on the surface but away from the polygon. Have him color as much as he can without crossing the polygon. Then have another student start coloring with another color at any previously uncolored place. Color as much as possible but do not cross the polygon. When the second student has colored as much as possible, the whole surface should be colored.

If you don't follow the instructions carefully you may get a polyhedron whose surface is not simple. Suppose, for instance, you fasten eight cubes together as in the drawing below. The polyhedron looks something like a square doughnut. Note that in fitting the eighth one, the intersection of the one you are adding with what you already have is the union of two faces which are not adjacent. The boundary (on the eighth cube) of the intersection is two simple closed polygons, not just one as it should be. There are many simple closed polygons on this surface which do not separate it at all. Some of these should be noted in class.

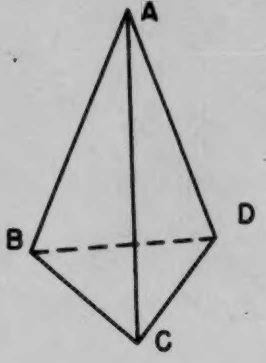


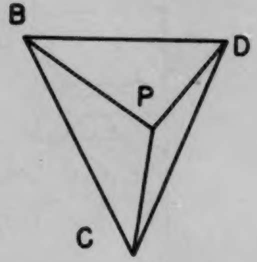
Exercises 8-7

1. Using a block of wood (with corners sawed off if possible), draw a simple closed polygon on the surface making it intersect most or all of the faces of the solid. Start coloring at some point. Do not cross the polygon. Color as much as you can without crossing the polygon. When you have colored as much as you can, start coloring with a different color on some uncolored portion. Again color as much as you can without crossing the polygon. You should have the whole surface colored when you finish.
2. Go through the same procedure as in problem 1 but with another 3-dimensional solid. Use one of your models or another block of wood. Make your simple closed polygon as complicated as you wish.

8-8. Counting Vertices, Edges, and Faces--The Euler Formula

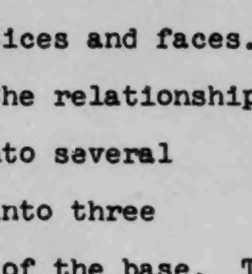
In section 8-6 you were asked to do some counting. We look at the problem in another way. A few of you may have discovered a relationship between F, E, and V. Consider the tetrahedron in the figure below. Its surface is a simple surface. What relationship can we find among the vertices, edges, and faces of it?





There are the same number of edges and faces coming into the point A, three of each. One may see that on the base there are the same number of vertices and edges. We have two objects left over: the vertex A at the top and the face (BCD) at the bottom. Otherwise we have matched all the edges with vertices and faces. So $F + V - E = 2$. Now let us ask what would be the relationship if one of the faces or the base were broken up into several 2-simplexes. Suppose we had the base broken up into three 2-simplexes by adding one vertex P in the middle of the base. The figure on the right above is the base with P in the middle. Our counting would be the same until we got to the base and we would be able to match the three new 1-simplexes which contain P with the three new 2-simplexes on the base. We have lost the face which is the base but we have picked up one new vertex P. Thus the number of vertices plus the number of 2-simplexes is again two more than the number of 1-simplexes. $F + V - E = 2$.

Next let us look at a cube. We have a drawing of one on the right. The cube has how many faces? How many edges? How many



vertices? Is the sum of the number of vertices and the number of faces two more than the number of edges? Let us see why this must be.

(1) The number of vertices on the top face is the number of edges on the top face.

(2) The number of vertices on the bottom face is the number of edges on the bottom face.

(3) The number of vertical faces is the number of vertical edges.

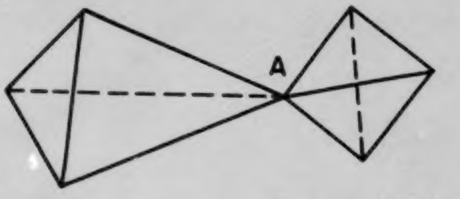
(4) All the vertices and edges are now used up. All the vertical faces are now used up. We have the top and bottom faces left.

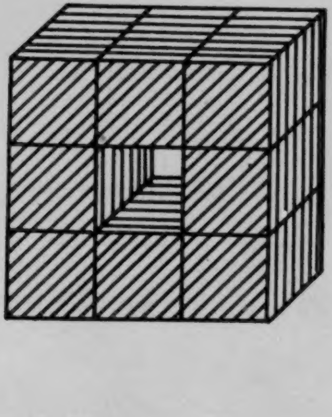
So F + V - E must be 2.

What would happen if each face were broken up into two 2-simplexes? For each face of the cube you would now have two 2-simplexes. But for each face you would have one new 1-simplex lying in it. Other things are not changed. Hence F + V - E is again 2.

Suppose we have any simple surface. Then do you suppose that V + F - E = 2? In the exercises you will be asked to verify this formula (which is known as the Euler Formula) in several other examples.

Let us now observe that the formula does not hold in general for surfaces which are not simple. Consider the two examples below.





In the figure on the left (the union of the two tetrahedrons having exactly the vertex A in common) V + F - E = 2. Count and see. Use models of two tetrahedrons if you wish. V + F - E should be 3. On each tetrahedron separately the number of faces plus the number of vertices minus the number of edges is 2. But the vertex A would have been counted twice. So V + F is one less than E + 4.

The figure on the right above is supposed to represent the union of eight solid cubes as in the last section. The possible ninth one in the center is missing. Count all the faces (of cubes), edges and vertices which are in the surface. For this figure V + F - E should be 0. (As a starter, V should be 32.)

Finally we put the Euler Formula in a more general setting. Suppose we have a simple surface and it is subdivided into a number (at least three) of non-overlapping pieces. Each of these pieces is to be bounded on the surface by a simple closed polygon. We think of F as the number of these pieces. We require that if two of these pieces intersect then the intersection be either one point or a polygonal path. The number E is the number of these intersections of pairs of pieces which are not just points. The number V is the number of points each of which is contained in at least three of these pieces. Then F + V - E = 2.

Exercises 8-8

1. Take a cardboard model of a non-regular tetrahedron. In each face add a vertex near the middle. Consider the face as the union of three 2-simplexes so formed. Give the count of the faces, edges, and vertices of 2-simplexes surface. How do the faces, edges, and vertices of this polyhedron compare with

those of the ployhedron you get by factoring four regular tetrahedrons to the four faces of a fifth?

2. Take a model of a cube. Subdivide it as follows. Add one vertex in the middle of each edge. Add one vertex in the middle of each face. Join the new vertex in the middle of each face with the eight other vertices now on that face. You should have eight 2-simplexes on each face. Compute F, V, and E. Do you get F + V - E = 2?
3. Make an irregular subdivision of any simple surface into a number of flat pieces. Each piece should have a simple closed polygon as its boundary. Count F, V, and E for this subdivision of the surface.

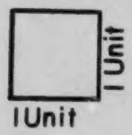
UNIT 9

VOLUMES AND SURFACE AREAS

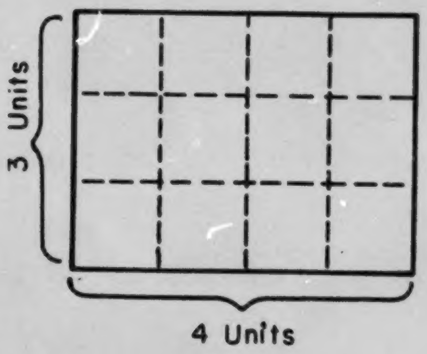
9-1 Areas of Plane Figures

In this unit we will be primarily concerned with surface areas and volumes of various figures. As a preliminary we will first discuss briefly areas of the interiors of certain simple closed curves. For some of you this will be a review, while for others it may be a first introduction to work with areas.

First let us look at the familiar figure of the rectangle and consider what we mean by the area of its interior. We choose for the unit of area, the region of a plane bounded by a square whose edges measure 1 unit.



The area of the interior of this square is 1 square unit. We may now determine the area of the interior of a rectangle the measurements of whose length and width are 4 and 3 units, respectively.

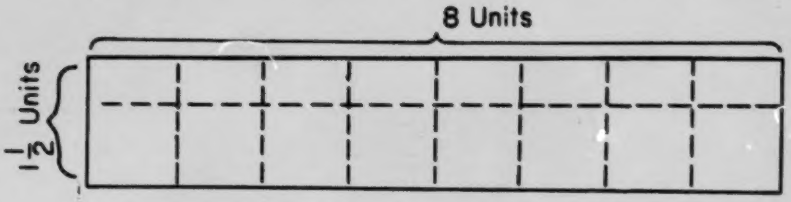


We can determine how many of our unit squares we need to fit the interior of the rectangle. In this case we see that we have 3 rows with 4 units to a row or 12 such units. We say then that the area of the interior of the rectangle is 12 of these suqare units.

Notice that we could have multiplied the number of rows by the number of squares in each row or the number of linear units in the length by the number of linear units in the width to determine the number of square units of Area.

Thus $A = \ell w$ (where A, ℓ and w are numbers)

In the case where the measurements of the length and width are 8 and $1\frac{1}{2}$ linear units respectively



we see that we could fit 8 full square units on the interior of the rectangle along the length. If we divide our square units in half so as to form two rectangles each 1 unit in length and $\frac{1}{2}$ unit in width we would need 8 of these rectangles to completely fill the interior of the given rectangle. Here too, the area is the number of square units that fill the interior. We see that there are 8 square units and 8 more halves of square units, a total of 12 square units of area.

We conclude that the area of the interior of the rectangle is 12 square units.

Once again we notice that the product of the numbers of linear units in the length and width is 12.

We assume that we can always find the area of the interior of a rectangle by fitting square units or fractions of square units on the interior of a rectangle:

Property 1. The number A of square units of area in the interior of a rectangle is the product of the numbers of linear units ℓ and w in the length and width respectively. Thus $A = \ell w$.

Since a square is a special kind of rectangle namely one whose adjacent sides are equal we have Property 2.

Property 2. The number of square units (A) in the area of the interior of a square whose edge is s units in length is the product of the numbers of linear units in its length (s) and its width (s). Thus $A = s \cdot s = s^2$.

As you probably know both the rectangle and square are special kinds of parallelograms.

The rectangle is a parallelogram whose adjacent sides are perpendicular. The square is a rectangle whose adjacent sides are equal.

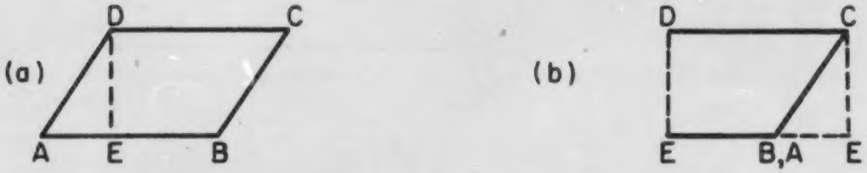


Figure 1.

In order to find the area of a parallelogram we can show that it can be changed to a rectangle as follows:

Consider parallelogram ABCD. Drop a perpendicular from vertex D to base $\overline{AB}$. Label the foot of the perpendicular E. This perpendicular is an altitude to the base $\overline{AB}$. Using a pair of scissors cut off right triangle ADE and place it as shown in figure 1b. It is clear that the rectangle formed has the same base and height as the original parallelogram.

Notice that for a rectangle the length of the base and the height are the same as the length and width. Since by Property 1 the number of square units A in the interior of the rectangle is ℓw, the number of square units in the area of the parallelogram is bh where b and h are the numbers of units in which base b and altitude h respectively.

Property 3. The number A of square units of area in the interior of a parallelogram is the product of the numbers of linear units b and h in the base and height. Thus $A = bh$.

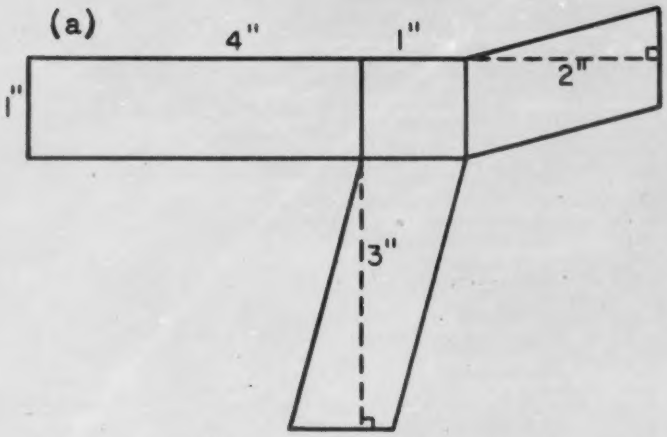
Exercise 9-1a

1. Find the areas of the interiors of each of the following:
   (a) Rectangle ABCD, where $\overline{AB}$ is 4 inches long and $\overline{BC}$ is 12 inches long.
   (b) Rectangle ABCD, where $\overline{AB}$ is 1/3 foot long and $\overline{BC}$ is 1 foot long.
   (c) Square ABCD where $\overline{AB}$ is 13 inches long.
   (d) Square ABCD, where $\overline{AB}$ is 1 1/12 feet long.
   (e) Parallelogram ABCD whose base $\overline{AB}$ is 6 inches long and whose height is 12 inches.

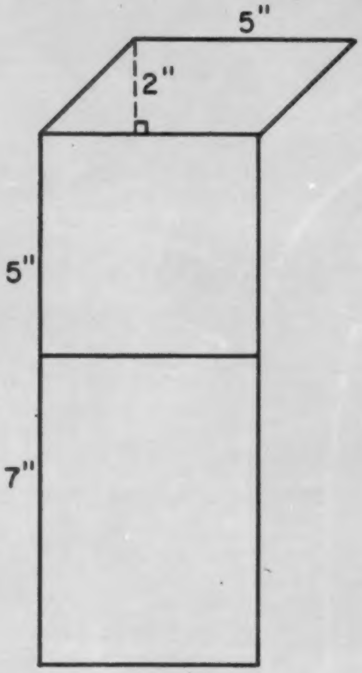
(f) Parallelogram ABCD whose base $\overline{AB}$ is $\frac{1}{2}$ foot long and whose height is 1 foot.

2. Determine the area of the interior of each of the following figures.

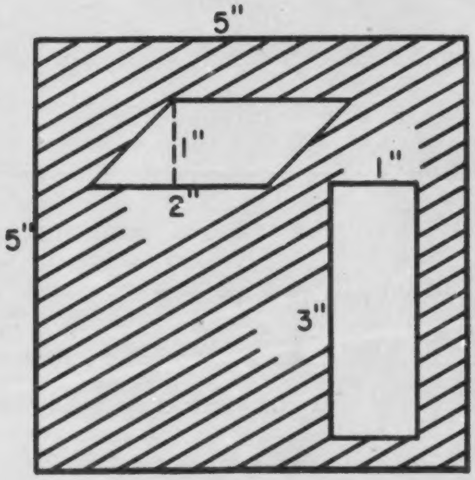
(a)



(b)



3. Find the area of the shaded portion.



The area of the interior of a triangle can be determined in a manner similar to that of a parallelogram.

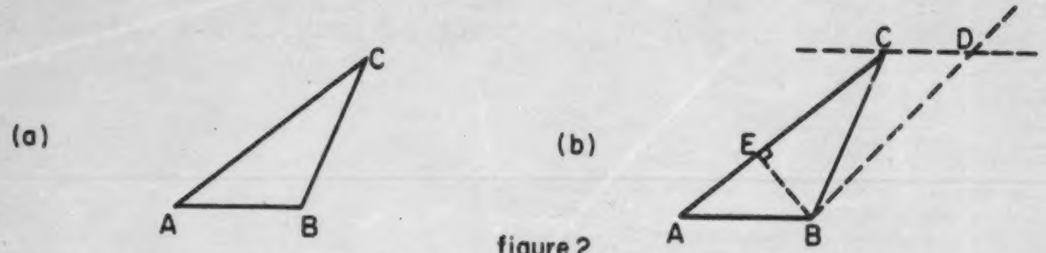
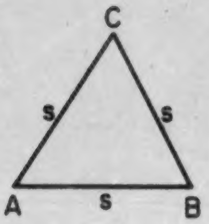


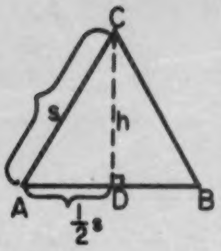
figure 2

Draw a line on the vertex of ∠C of △ABC (see figures (2a) and (2b) above) parallel to the base $\overline{AB}$. Draw a line on the vertex of ∠B parallel to side $\overline{AC}$. Label the point of intersection of the lines D. We now have a parallelogram ABDC. The diagonal $\overline{BC}$ of the parallelogram divides it into two congruent triangles ABC and BCD which therefore have the same areas. Since by Property 3 the number of square units in the area of the interior of a parallelogram is bh ($\overline{AC}$ and $\overline{EB}$ are the base and altitude respectively), b and h are the numbers of linear units in them and since the parallelogram is composed of the two congruent triangles ABC and BCD with base and height equal to those of the parallelogram, it follows that the area of the interior of the triangle ABC is $\frac{1}{2}$ that of parallelogram ABDC. Thus.

Property 4. The number (A) of square units of area in the interior of a triangle is $\frac{1}{2}$ the product of the numbers of linear units of measure in the base and height. $A = \frac{1}{2}bh$.

An equilateral triangle is a triangle all of whose sides are of equal measure. The number of square units in the area of the interior of this triangle is $\frac{1}{2}bh$ (Property 4). Since we will have occasion to work with equilaterial triangles we will apply Property 4 to determine the area of the interior of this special triangle.





In the figure triangle ABC is an equilateral triangle whose sides are s units in length. Drop a perpendicular from the vertex C to side (base) $\overline{AB}$ and label the foot of the perpendicular D. $\overline{CD}$ is the altitude and h is units in length. It can be shown that $\overline{CD}$ bisects the base $\overline{AB}$ (divides $\overline{AB}$ into two equal segments of length $\frac{1}{2}$). Hence by the Pythagorean property we have:

$$s^2 = h^2 = (\tfrac{1}{2}s)^2 \quad \text{or}$$

$$h^2 = s2 - (\tfrac{1}{2}s)^2 \quad \text{or}$$

$$\text{now} \quad h^2 = s^2 - \tfrac{1}{4}s^2$$

$$\text{and} \quad h^2 = \sqrt{\tfrac{3}{4}s^2}$$

$$\text{also} \quad h = \sqrt{\tfrac{3}{4}s^2}$$

$$\therefore \quad h = \frac{\sqrt{3}}{4}\, s$$

The last step in the simplification is probably unfamiliar to you. Such work with square roots will be discussed in the 9th grade, but we will assume the result here. We could check that $\frac{\sqrt{3}}{2}s$ is the square rost of $\frac{3}{4}s^2$ in the following way.

$\left(\frac{\sqrt{3}}{2}\cdot s\right)\left(\frac{\sqrt{3}}{2}\cdot s\right) = \left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right)$ (ss) by the associative and commutative laws of multiplication.

$$\left(\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}\right) (ss) = \frac{\sqrt{3} \cdot \sqrt{3}}{4} \cdot s^2$$

Now $\sqrt{3} \cdot \sqrt{3}$ is 3 because of the way in which we defined square root.

Hence: $\sqrt{\frac{3}{4} \cdot s^2} = \frac{\sqrt{3}}{2} \cdot s$, since $\left(\frac{\sqrt{3}}{2} \cdot s\right)\left(\frac{\sqrt{3}}{2} \cdot s\right) = \frac{3}{4}s^2$.

Now since the number (a) of square units of area in the interior of a triangle is $\frac{1}{2}bh$ we have $A = \frac{1}{2}(s) \left(\frac{\sqrt{3}}{2} \cdot s\right)$

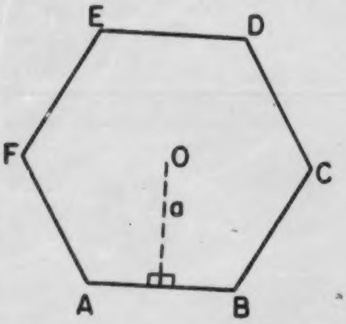
Property 5. The number (a) of square units of area in the interior of an equilateral triangle is $A = \frac{\sqrt{3}}{4}s^2$.

You may recall that $\sqrt{3}$ is approximately 1.732 so that $\frac{\sqrt{3}}{4} \approx 0.433$. The approximate value of A is therefore as follows:

$$A \approx (0.433)s^2$$

A regular polygon is defined to be a polygon whose sides are equal in measure and whose angles are equal in measure. The area of a regular polygon can be found as follows:

(a)



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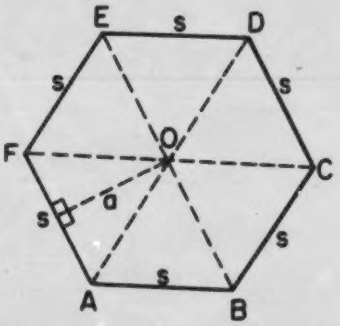


Join the center of the regular polygon to each vertex of the polygon. (The center is the point on the interior which is equally distant from the verticles of the polygon) Thus if there are n verticles there will be n congruent triangles.

The figure is an illustration of the case where $n = 6$. The area of the interior of the regular polygon is the sum of the areas of the interiors of the n equal triangles into which the regular polygon has been divided. All the triangles have equal bases (sides of the regular polygon) and all the triangles have equal altitudes (the center of the polygon is equally distant from each side and this distance is the length of the altitude, abbreviated a).

The area of the interior of the regular polygon of n sides is the sum of the areas of the interiors of the n congruent triangles with base s the measure of length of the side of the regular polygon, and height a,

Thus.

$$A = n \cdot (\tfrac{1}{2}as) = \tfrac{1}{2}a\,(ns)$$

But ns is the perimeter of the regular polygon (the sum of the n sides). $\therefore$

Property 6. The number (A) of square units of area in the interior of a regular polygon is $\frac{1}{2}$ the product of the numbers of linear units in its perimeter and the distance from the center to each side $A = \frac{1}{2}ap$.

Exercises 9-1b

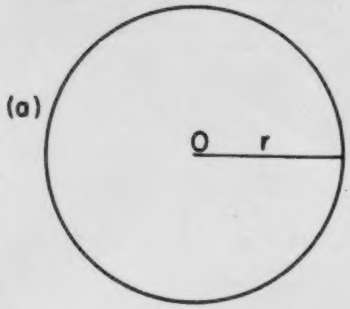
1. Compute the areas of the interiors of each of the following:
   (a) Triangle ABC whose base is 5 inches long and whose altitude is 7 inches long.
   (b) Triangle ABC whose base is 15 feet long and whose altitude is 30 feet long.
   (c) Equilateral triangle ABC whose side is 16 feet long.
   (d) Equilateral triangle ABC whose side is 8 yards long.
   (e) Regular hexagon ABCDEF whose center is 17.3 inches from the sides and whose side is 20 inches long.

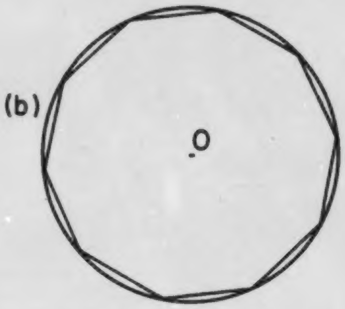
(f) Regular pentagon ABCDE whose center is 27.5 inches from the sides and whose side is 40 inches long.

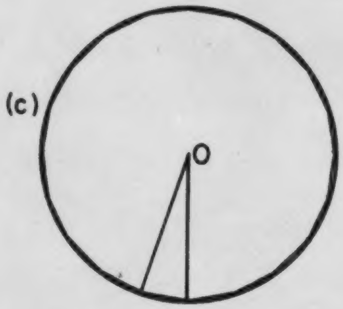
(g) Regular octagon ABCDEFGH whose center is 72.5 feet from the sides and whose side is 60 feet long.

(h) Regular decagon ABCDEFGHIJ whose center is 30.8 inches from the sides and whose side is 20 inches long.

Consider circle O in the figures below.







We say that a regular polygon of n sides is inscribed in a circular region if the verticles of the regular polygon are points on the circle. It is clear from the figures above that the more sides the inscribed polygon has the shorter will be the length of each side. It is also clear that as n gets larger and larger it will be more and more difficult to distinguish between the regular polygon and the circle.

We could say that the area of the interior of the inscribed polygon is approximately equal to the area of the interior of the circle. It will always be less than the area of the interior of the circle since there will always be points on the circle that are not verticles of the inscribed regular polygon. Therefore there is always some portion of the area of the circle which is not contained in the interior of an inscribed regular polygon.

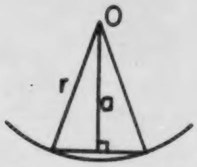
However, for large values of n the areas are very close and we can think of the area of the interior of the circle as the upper limit of that of the inscribed regular polygons. We could say that:

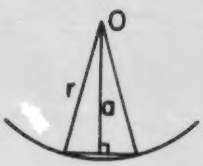
(Area of interior of polygon) < (Area of interior of circle).

It should also be noted that the larger we take n (the number of sides) the closer is the measure of the distance of the center of the inscribed polygon to a side to the radius of the circle. Likewise the closer is the perimeter to the circumference of the circle. Now we have seen that the number of square units of area in the interior of the polygon is $\frac{1}{2}ap$. But we have just observed that when n gets very large a gets close to r and p gets close to $2\pi r$ so we are led to conclude.

Property 7. If r is the number of linear units in the radius of a circle, and (A) the number of square units of area in its interior, then $A = (\frac{1}{2}r)(2\pi r)$ or

$$A = \pi r^2$$





Exercises 9-1c

1. Compute the area of the interior of each of the following circles. The measurements in each case are in inches.

(a) $r = 5$

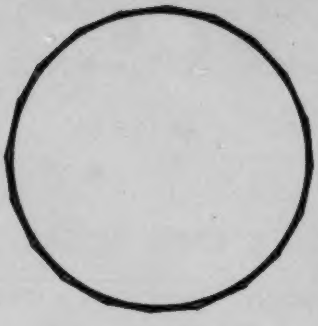
(b) $r = 10$

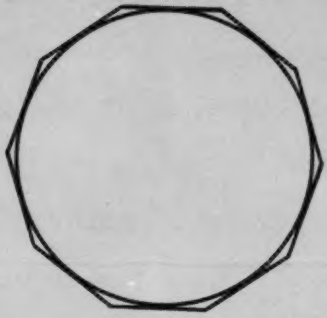
(c) $r = 20$

(d) $r = 4\frac{1}{2}$

(e) $d = 20$

(f) $d = 15$





2. Construct models 1, 2, 3, 4, 5 (see end of unit.)
3. By examining the results of parts (a), (b), (c) in Problem 1 above tell the effect on the area of a circle if its radius is doubled.

4. (a) BRAINBUSTER. Imagine that you have inscribed a regular polygon of 20 sides in a circle and that you have divided this polygon into 20 congruent triangles by joining its center to each vertex. Show that these triangles can be rearranged into a parallelogram whose height is almost the radius of the circle and the length of whose base is almost one half the circumference of the circle.

(b) BRAINBUSTER. Imagine that you have circumscribed a regular polygon of n sides (n very large) about a circle o. (This means that each side of the regular polygon contains exactly one point of the circle. Develop a plausible argument to support the following statement:

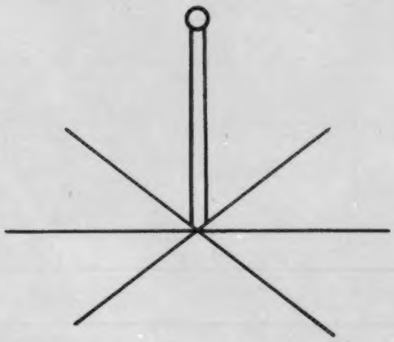
$$\left\{\begin{matrix}\text{Area of} \\ \text{interior of circle}\end{matrix}\right\} < \left\{\begin{matrix}\text{Area of interior of} \\ \text{circumscribed polygon}\end{matrix}\right\}$$

Together with our discussion above this would show that

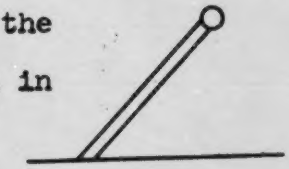
$$\left\{\begin{matrix}\text{Area of interior of} \\ \text{circumscribed polygon}\end{matrix}\right\} < \left\{\begin{matrix}\text{Area of} \\ \text{interior of circle}\end{matrix}\right\} < \left\{\begin{matrix}\text{Area of interior of} \\ \text{circumscribed polygon}\end{matrix}\right\}$$

9-2 Planes and Lines

Before going on, let us review briefly some of the simple ideas about planes and lines. You are already somewhat familiar with parallel planes. These are planes which do not have any points in common, that is, whose intersection is the empty set. Such a pair of planes is suggested by the floor and ceiling of your classroom, or by different floors of an apartment house, or by the covers on a book when the book is closed. Find at least five examples of pairs of parallel planes suggested by things in your classroom.

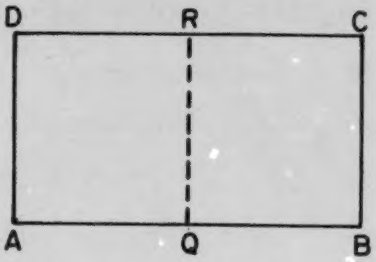


Imagine a flagpole standing in the moddle of a level playground, and think of the lines on the playground which run through the base of the pole as shown. What relation does there appear to be between the line represented by the flagpole and these lines drawn on the playground? Our experience certainly suggests that the pole is perpendicular to each of these lines. In fact, if it were not, then by standing in certain positions the pole would appear like this, which is not at all in accord with our observation. We describe this relationship by saying that the pole is perpendicular to the playground. In general a line which meets a plane in a point A is said to be perpendicular to the plane if it is perpendicular to every line in the plane through A . If a segment lies on a line perpendicular to a plane, we will say

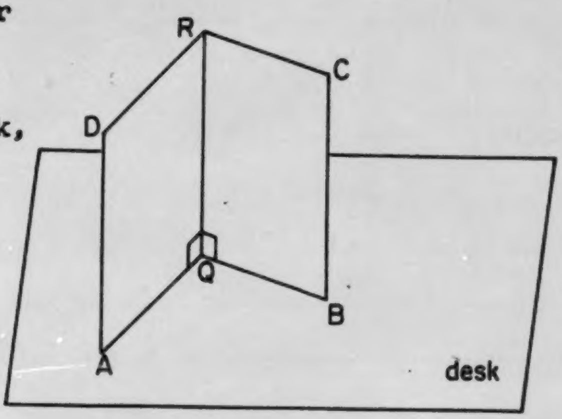


that the segment is perpendicular to the plane.

Now try the following simple experiment. Take a piece of notebook paper as shown and fold it over so $\overline{AD}$ falls on $\overline{BC}$. The crease you have made is represented by the dotted segment $\overline{QR}$. Then $\angle AQR$ and $\angle BQR$ are both right angles.



How do you know? Now take the paper and set it on your desk as show, in the position of a partly opened book, so that segments $\overline{AQ}$ and $\overline{BQ}$ lie on the plane of the desk top. Would you agree that QR is now perpendicular to the desk top? If so, notice that you have found a line perpendicular to a plane by making it perpendicular to just two different lines in the plane. This illustrates the following property of perpendiculars.



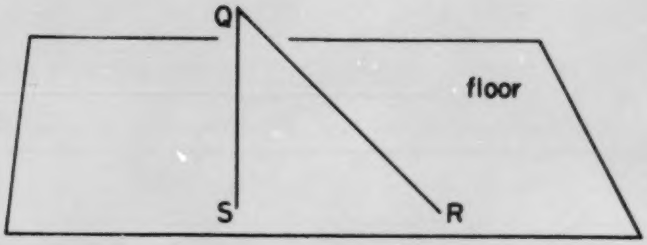
Property 8. If a line is perpendicular to two distinct intersecting lines in a plane, it is perpendicular to the plane.

The next time you help to put up a Christmas tree, check to see whether or not it is perpendicular to the floor by seeing if it is perpendicular as viewed from two different points not in the same line from a point on the tree. If so, it is perpendicular from all points of view. This is an application of Property 8.

As another example, examine Model 5 and look at one of the segments which connect a vertex of one hexagonal end with a vertex of the other. As you see, this segment is a part of two rectangles. It is therefore perpendicular to two segments in each hexagon. By Property 8 the segment is therefore perpendicular to the planes of both hexagons. Examine Model 4 similarly and satisfy yourself in the same way that every edge of the solid is perpendicular to the planes of two of its rectangular faces. What about the line where two walls of your classroom meet? What relation does it have to the planes of the ceiling and floor?

Examine Model 7 to satisfy yourself that the result actually applied to this also.

Now try another experiment. Tie one end of a string to some convenient point Q in your classroom which has a clear space below it. If nothing else is available tie it to a yardstick placed over the back of a couple of chairs, and have someone hold the ends so they won't move. Now select a point R on the floor and notice how much string it takes to join Q to R. By varying R try to find the point S on the floor which requires the least amount of string. When you have located the point S , notice the position QS of the string. What relation does it seem to have to the floor? Would you agree with the following statement?



Property 9. The shortest segment having one end at a given point Q outside a plane and the other on a given plane r not containing the point Q is a segment perpendicular to the plane r .

This shortest distance is called the distance from Q to r .

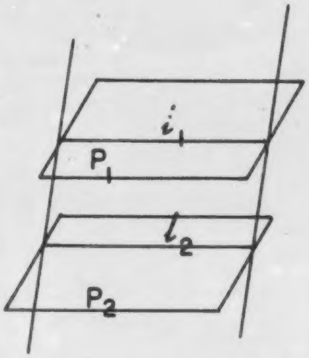
Imagine now several nails in the ceiling of your room, to each of which is attached a string. In each case the string is then attached to the nearest point of the floor as in our experiment above. What do you expect about the lengths of the different strings? Will they be all the same? This illustrates the following fact

Property 10. If two planes are parallel the distances from different points of one plane to the other plane are all the same.

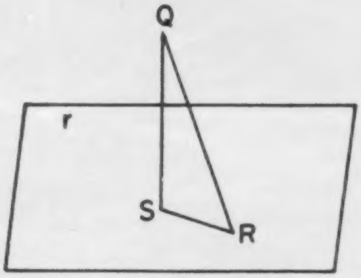
The constant distance in Property 10 is called the distance between the parallel planes. Actually the segments involved in Property 10 are perpendicular to both planes. We have already noticed this for the lateral edges of a right prism.

Exercises 9-2

1. Give five examples of pairs of parallel planes with lines perpendicular to both planes of the family.
2. Make Models 9 and 10.
3. Make Models 11 and 12.
4. If two parallel planes P_1 and P_2 are intersected by a plane r in lines ℓ_1 and ℓ_2, then ℓ_1 must be parallel to ℓ_2. Explain why this is true.



5. We actually could have proved Property 9 instead of observing it by experiment. Give the reasons in the following proof.



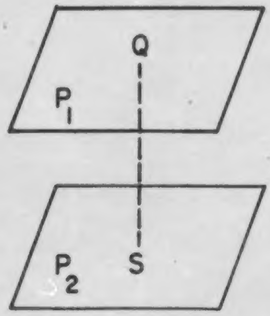
Let S be the point of r so that $\overline{QS}$ is perpendicular to r . Draw segment $\overline{SR}$.

(a) $\angle QSR$ is a right angle. Why?

(b) $\overline{QR}$ is the hypotenuse of a right triangle. Why?

(c) $\overline{QR}$ is longer than $\overline{QS}$. Why?

But since R was any point of r except S , this shows that $\overline{QS}$ is the shortest segment.



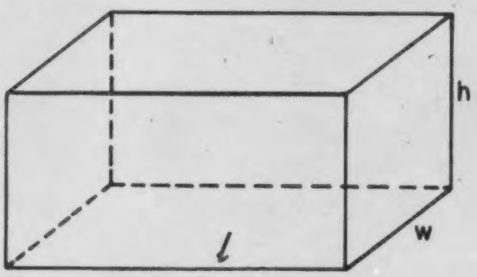
6. **Brainbuster**. Give the following proof. A segment $\overline{QS}$ has its ends on the parallel planes p_1 and p_2 . If $\overline{QS}$ is perpendicular to p_2 , show it must also be perpendicular to p_1 .

[Hint: Draw a couple of planes through $\overline{QS}$.]

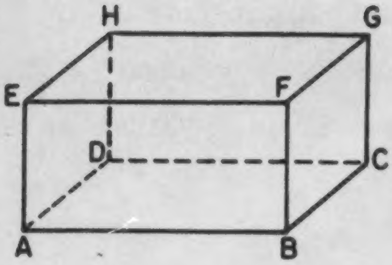
9-3 Right Prisms

We have seen that area is a measure of the region of a plane bounded by a simple closed curve. Volume is a measure of a region of space.

Consider a rectangle in a plane whose length and width are l and w linear units respectively. Imagine that we lift this rectangle perpenticularly to a parallel plane at a distance of h linear units above the given plane. If we join the vertices of the elevated rectangle to the corresponding vertices of the original one, we have a right rectangular prism, sometimes called a rectangular solid.



The right rectangular prism is a surface consisting of the interiors of the given rectangles together with the interiors of the rectangles formed by the segments joining corresponding vertices, and the segments bounding these interiors. The interiors are called the faces of the prism, the bounding segments are its edges, and points where two or more edges intersect are vertices. The interiors of the original rectangles are called bases, and the other faces are called the lateral faces of the prism. The segments joining corresponding vertices of the bases are called lateral edges.



You will notice that in a figure of this kind all of the faces are rectangles, so actually any pair of parallel faces can be considered as the bases of the figure. If, in the above figure, rectangles ABCD and EFGH are considered as the bases, then;

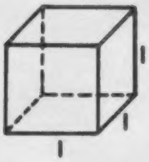
rectangles ABFE, BCGF, DCGH, and DAEH are the lateral faces

segments $\overline{AE}$, $\overline{BF}$, $\overline{CG}$, $\overline{DH}$ are the lateral edges

points A, B, C, D, E, F, G, H are the vertices.

In Model 4 point out the faces, edges, and vertices. How many are there of each? If the small squares in this model are taken as bases, point out the lateral faces and lateral edges.

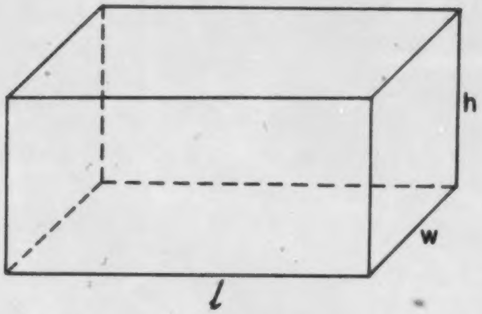
It is natural to take as a unit of volume the interior of a unit cube, that is a cube each edge of which is one unit of length.



Examine Models 1, 2, and 3. These are models of the cubic inch, the half cubic inch, and the quarter cubic inch. For each linear unit of measure there is a corresponding cubic unit of volume, as the cubic foot, cubic yard, cubic centimeter, etc.

The surface area of a unit cube is the sum of the areas of the interiors of its faces. Thus the surface area is 6 square units since there are six congruent faces and the area of the interior of each face is 1 square unit. Observe the six faces on Model 1. Similarly, the surface area of any rectangular right prism is the sum of the areas of the interiors of all its faces.

The measures of the edges in the three different directions are often called the length, width, and height. In the figure below ℓ, w, and h are the numbers of linear units of measure in the length, width, and height, respectively.



Notice there are two rectangles the areas of whose interiors are ℓw square units, two the areas of whose interiors are $w h$ square units, and two the areas of whose interiors are ℓh square units. Thus:

Property 11. The number of square units of surface area in the interiors of the bases and lateral faces of a right rectangular prism is $S = 2[\ell w + \ell h + wh]$

What is the surface area of model 4?

The measure of the volume of any region in space is the number of unit cubes (or part unit cubes) of volume necessary to fill it.

The volume of the interior of the right rectangular prism can be determined in a fashion similar to the way in which we determined the area of the interior of a rectangle. Consider a right rectangular prism whose length, width, and height measure 2, 3, and 4 linear units respectively.

We can take unit cubes such as figure 2b and place them in the interior of the right rectangular prism until we completely cover the area of the interior of the base. It is clear that we will need 6 such cubes to cover the rectangular base. We can now place a second layer of 6 cubes on top of the first and so on. Thus we see that we will need 4 layers with 6 cubes in each layer, or 24 such unit cubes. We say that the volume of the interior of the right rectangular prism is 24 cubic units.

Notice that if we take the product of the number of square units of area in the interior of the base and the number of linear units of measure in the height we obtain $4(2 \times 3)$ or 24, the number of cubic units of volume of the interior of the right rectangular prism. If we let B stand for the number of square units of area in the interior of the base and let h stand for the number of linear units in the height and let V stand for the number of cubic units of volume in the interior of the right rectangular prism we have: $V = Bh$.

If the length, width, and height of the right rectangular prism has been 2, 3, and $3\frac{1}{2}$ linear units, we would simply partly fill the right rectangular prism with 3 layers of unit cubes, 6 cubes to a layer. The remaining half layer can be filled by dividing 3 unit cubes into 2 equal parts and placing the 6 right rectangular prisms whose length, width and height measure 1, 1, and $\frac{1}{2}$ linear units respectively on top of the last layer. We now can say that the interior of the right rectangular prism is completely filled or that the number of cubic units in the volume of the interior of the right rectangular prism is $(6 \times 3) + (6 \times \frac{1}{2})$ or $18 + 3$ or 21. Notice here too if we take the product of the number of square units of area (B) in the interior of the base and the number of linear units (h) in the height we obtain the same number (V) of subic units in the volume of the interior of the right rectangular prism, namely, $6 \times (3\frac{1}{2})$ or 21.

It seems clear that if there are B square units of area in the base of a right rectangular prism and h linear units in its height, the interior can be filled with h layers each with B unit cubes so the total unit cubes used should be Bh. This leads us to state the following property.

Property 12. The number (v) of cubic units of volume in the interior of a right rectangular prism is the product of the number (B) of square units of area in the interior of the base and the number (h) of linear units in the height. Thus; $V = Bh$.

Since the cube is a special kind of right rectangular prism, namely one whose length, width, and height are all equal in length (s units long) we have the following property.

Property 13. The number of cubic units of volume V in the interior of a cube whose edges are s linear units in length is given by

$$V = s^2 \cdot s = s^3$$

Since the cube has 6 faces whose interiors are equal in area, namely, s^2 square units we have.

Property 14. The number of square units S of surface area in the faces of a cube is $6\,s^2$.

Exercises 9-3a

1. Make models, using your own patterns for rectangular right prisms whose measurements in inches are as shown. You may find it helpful to look at the pattern for Model 4.

   (a) $\ell = 1, \quad w = 2, \quad h = 2$

   (b) $\ell = 2\frac{1}{2}, \quad w = 2, \quad h = 2$

   (c) $\ell = 1\frac{1}{2}, \quad w = 1\frac{1}{2}, \quad h = 2$

2. Take copies of Models 1, 2, 3 of the cubic inch, cubic half inch and cubic quarter inch and assemble them into models like the three models in problem 1. Get the volumes by counting up the number of unit cubes (and parts of them) used. For this problem you will need to pool your models with those of some of your classmates.

3. Compute the volume of each right rectangular prism constructed in problem 1 using Property 12.

4. Compare your results of problems 2 and 3.

5. Calculate the surface area of each right rectangular prism constructed in problem 1 using Property 11.

6. Construct Models 6 and 7.

7. Construct Model 8.

General Right Prisms

You have already learned about right rectangular prisms. Now we shall look at some general right prisms. Take models 1 through 7 and examine them carefully. They are all examples of right prisms. How do they resemble each other? How do they differ? Keep them in mind as you read the following discussion.

Imagine two cosgruent polygons (i.e. polygons with the same size and shape) so placed in parallel planes that when the segments are drawn joining corresponding vertices of the polygons the quadrilaterals formed are all rectangles. The figure formed is called a right prism. The prism consists of the interiors of the rectangles and the interiors of the original polygons together with the segments which bound these interiors. As in the case of the right rectangular prism, the segments are its *edges*, and the points where two or more edges meet are *vertices*. The interiors of the original polygons are called the bases of the prism and the other faces its *lateral faces*. Similarly the segments joining corresponding vertices of the two bases are called *lateral edges*.

A prism is triangular, square, and so on according as its bases are triangles, squares, and so forth.

Referring to Section 9-2 you will see that for any right prism the lateral edges are perpendicular to the planes of the bases.

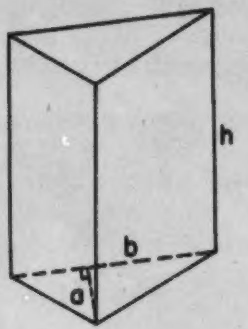
Class Exercises

(a) Which of the models are right rectangular prisms?

(b) Which are right triangular prisms?

(d) Which have hexagons for faces?

Consider a triangular right prism

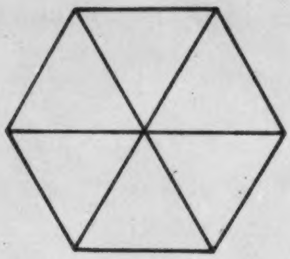


The number b of square units in the interior of the triangular base is $\frac{1}{2}$ ba where b and a are the numbers of linear units in the base and altitude of the triangle. (Notice that letter a has been used for the height of the triangle to avoid confusion with the height of the prism itself.)

Let h be the number of linear units in the height of the prism. Carefully fill your unit cube with salt and pour the contents into model 6. Continue this procedure until the model is completely filled with salt. Keep count of the number of cubic units of salt poured. This is the number of cubic units of volume for the model.

Now let us calculate the number of square units of area in the base of Model 6. Notice from the pattern for Model 6 that the base is the interior of a right triangle so it will be easy to find the area. Call this number of square units B . What do you get for B ? Find the product of B and the number h of linear units in the height of the model. How does this agree with the volume you just obtained?

Do the same for Models 4, 5, and 7. In the case of Model 5 the base is a regular hexagon with each side 1 inch long. A regular hexagon is made up of six equilateral triangles as shown



so the area of the interior of the hexagon may be found from the area of the interior of an equilateral triangle discussed above. Similarly, the base of Model 7 is formed from the interiors of two equilateral triangles as shown, so its area is easily found. In each of these cases how does the calculated product Bh compare with the volume as found by pouring salt? Do you find them approximately equal? (Keep the figures on these volumes for future reference.)

Even if you have done your work carefully there probably will be some slight discrepancy between the measured volume and the computed product. This experimental work leads us to state the following property.

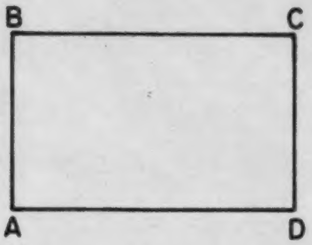
Property 15. In general the volume (V) of the interior of a right prism with polygonal bases is the product of the number of square units (B) in the interior of the base and the number of linear units (h) in the height.

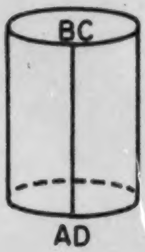
$$\text{Thus} \quad V = Bh.$$

We should expect these results if we recall how we originally defined volume. We could have placed layers of cubic inches and fractions of cubic inches on top of each other until the interiors of the various right prisms were completely filled. The number of cubic units in the volume should then be the product of the number B of cubic units in each layer and the number n of layers.

Right Circular Cylinders

Consider a rectangle in a plane (Figure 3a)





Imagine that we lift this rectangle out of the plane and place it perpendicular to the plane. Now imagine that we "bend" it in a circular fashion till the edges AD and BC coincide and the bases of this figure are circles as shown in Figure 3b. The figure we have described is a right circular cylinder. Its lateral surface has the same measure as the interior of the original rectangle. Its bases are the interiors of circles whose circumferences are the length of the original rectangle. The height of the cylinder is the height of the original rectangle.

Class Exercise

Carefully fill model 8 with cubic inches of salt until it is completely filled up. Be sure to keep a count of the total number of cubic inches of salt you needed. Now calculate the area of the interior of the base of the cylinder. For this notice the radius in the pattern of Model 8. Multiply this number by the number of linear units in the height of the cylinder. Compare the product with the number of cubic inches of salt used to fill the cylinder.

Even if you have been careful in your work there will be some slight discrepancy between the two results, but the experiment leads us to state the following property.

Property 16. The number V of cubic units of volume in the interior of a right circular cylinder is the product of the number of square units (B) in the interior of the circular base and the number of linear units (h) in the height. Thus $V = \pi r^2 h$.

We could have anticipated this result if we recall how we originally defined volume. We could have placed layers of cubic inches and fractions of cubic inches on top of each other until the interior of the cylinder was completely filled. The number (V) of subic inches would then be the volume of the interior of the cylinder.

As in the case of the prism it seems clear that V should be the product of the number of cubes B in each layer and the number h of layers.

The following property relating to the total surface area may be verified by looking at the three pieces of which the surface consists.

Property 17. If r and h are the numbers of linear units in the radius of the base and the height of a right circular cone and S the number of square units in the total area, then

$$S = 2\pi rh + 2\pi r^2 \quad \text{or} \quad 2\pi r(r + h)$$

Actually $2\pi rh$ is the area of the lateral surface. Why? Also $2\pi r^2$ is the area of the circular bases.

Exercises 9-3b

1. (a) Refer to patterns for Models 4, 5, 7, 8 and find the perimeters of the bases. In the case of Model 4 consider the small squares as bases.
   (b) Are these perimeters all equal to each other?
   (c) You have already found the volumes of these four models. Refer to your previous work and write the results here.
   (d) Are the volumes equal?
   (e) List the models in order of their volumes from smallest to largest.
   (f) On the basis of your experience in this problem, what conjecture would you make about the area of the interior of a circle as compared with those of polygons whose perimeters equal the circumferences of the circle?
2. (a) When you found the volumes of Models 4 and 6, did you find them equal?
   (b) Check part (a) by filling one with salt and pouring it into the other.
   (c) Find the perimeters of the bases of these models. Are the perimeters equal.

9-4 General Prisms

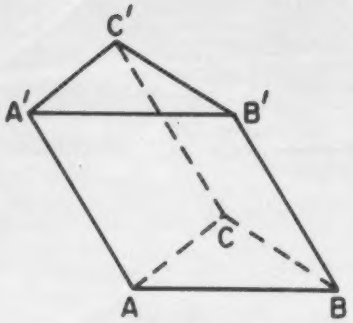
You have already learned about right prisms. Now we look at some more general figures. Take Models 6, 11, and 12, and examine them carefully. As you know Model 6 is a right prism, actually a triangular one. In what ways do Models 11 and 12 differ from Model 6? Now examine Model 7 which is a right prism and the related Models 9 and 10. How do Models 9 and 10 differ from Model 7?

All of these models represent figures called prisms. Keep them in mind as you read the following definitions.

Imagine two congruent polygons (i.e. polygons with the same size and shape) so placed in parallel planes that when the segments are drawn joining corresponding vertices of the polygons the quadrilaterals formed are all parallelograms. The figure formed is then called a prism. The prism consists of the interiors of these parallelograms and the interiors of the original polygons together with the segments which bound these interiors.

As in the case of right prisms these interiors are called faces of the prism, the segments are its edges, and the points where two or more edges meet are vertices. The interior of the original polygons are called the bases of the prism and the other faces its lateral faces. Similarly the segments joining corresponding vertices of the two bases are called lateral edges.

For example, in the accompanying figure the bases are the interiors of triangles ABC and A'B'C, the lateral faces are interiors of the parallelograms ABB'A', CBB'C', CAA'C', and the lateral edges are segments $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$.



As you know if the parallelograms determining the lateral faces are all rectangles, as in Models 5 and 6, the figures are right prisms. Other prisms like Models 11 and 12 are often called oblique prisms because their lateral edges are not perpendicular to the planes of the bases. They seem to lean to one side.

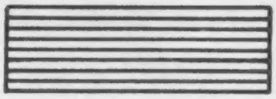
In Models 6 and 11 point out the bases, lateral faces and lateral edges.

Now do the same for Models 7 and 9. In these last cases did you have any difficulty identifying the bases? How did you decide? The difficulty here illustrates an interesting property of Models 4, 7, 9, 10. In these models all faces are parallelograms. (recall that a rectangle is a special case of a parallelogram.) In these figures any pair of opposite faces may be considered the bases and the other faces are then the lateral faces. Such figures can really be thought of as prisms in three ways. Because their faces are all parallelograms, such prisms are given the mouth-filling name parallelepipeds. The rectangular prisms which you studied earlier are the special parallelepipeds where all the faces are rectangles.

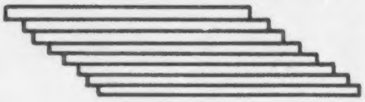
Suppose we ask ourselves now about finding the volumes of the interiors of oblique prisms. It isn't so easy as for the right prisms to imagine filling up the interior with cubes and parts of cubes of volume. With the lateral edges not perpendicular to the bases the cubical blocks don't fit nearly without a lot of messy patch-work. Of course we can take our cubic inch model as a container and see how many cubic inches of salt it takes to fill each of our models. In fact there is an interesting project which you may like to carry out, but what we would really like is some way of finding the volume from measurement of the figure as we did for the right prisms.

In the process of moving the cards around for any particular pack two things are unchanged. One is the size and shape of the cards, the other is the thickness of the pack. Thus all the different prisms which arise have congruent bases and also have equal distances between the planes of the two bases. To describe this we introduce the term altitude. Any segment perpendicular to the planes of the bases of a prism and having one end on each is called an altitude of the prism. From the last section we know that all these altitudes have the same length. The length of an altitude is called the height of the prism. Thus all the different prisms arising from the same pack of cards have congruent bases and equal heights.

Now in pushing the cards around we have clearly not changed the amount of cardboard present. It is then tempting to conclude that all the different prisms we get from the same stack of cards have the same volumes. However, let us be a little critical. If



we imagine the cards perfectly made, then in their original vertical position they would fit together perfectly like this illustration.



But as soon as we push them out of line the edges of the cards no longer fit smoothly but look like the second illustration instead. You can easily feel the effect by running your fingernail over the edges, and it can be apparent to the eye also if the stack of cards gets far out of the vertical. Still the irregularities seem to be rather small, especially if we imagine we have very thin cards, perhaps made of tissue paper.

At least we seem to have basis for making the following conjecture (Conjecture is a big word for what we hope is an intelligent guess). Conjecture: If two prisms have congruent bases and equal heights, they have equal volumes.

To test this conjecture look at Models 6, 11, and 12. Do they appear to have congruent bases? (Try putting the bases against each other to see.) Do they have equal heights? For this it may help to stand them on their bases and lay a ruler across their upper bases to see if it seems level. Do you agree that these models have congruent bases and equal heights? Now fill Model 6 with salt and pour into Model 11. Did you have too much salt or not enough, or did it seem to be just right? (This sounds like the three bears!) Do your results on this experiment confirm the conjecture above?

Carry out the same experiment with Models 7, 9, and 10.

(For this experiment treat the small parallelogram as the bases since otherwise you do not get congruent bases.) Does the result confirm the conjecture?

Since the conjecture seems to be borne out in practice, we will list it now as a property.

Property 18. If two prisms have congruent bases and equal heights, they have equal volumes.

From Property 18 the volume of any prism is the same as that of a right prism whose base is congruent to the given one and having the same height. But since by Property 15 we know how to find the volume of the right prism, we obtain at once the following Property.

Property 19. The number of cubic units of volume in the interior of any prism is obtained from the formula

$$V = Bh$$

where B is the number of square units of area in its base and h the number of linear units in its height.

For example, the base of Model 11 is a right triangle with legs having lengths approximately 2 inches and $2\frac{1}{4}$ inches. Check these measurements on your model. The number B of square inches in the area of the base is therefore

$$B = \frac{1}{2}(2)(2\frac{1}{4}) = \frac{9}{4}$$

so the area of the base is $\frac{9}{4}$ sq. in. Why? You should find the height is 4 in. (Note this is not the same as the length of the lateral edge which is about $4\frac{1}{8}$ in. Thus $h = 4$

$$V = (\frac{9}{4})4 = 9$$

so the volume is 9 cu. in.

Exercises 9-4

1. Check the accuracy of the last calculation by taking your cubic inch measure, Model 1 and see if 9 fillings of it will just fill Model 11.

2. Make Models 13 and 14.

3. Make Models 15, 16, and 17.

4. Is a lateral edge of a right prism an altitude of the prism? Why?

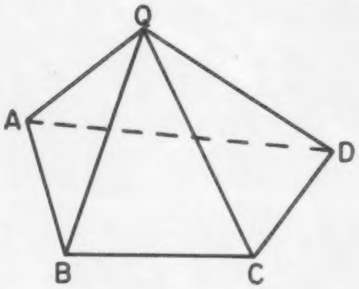
5. Is a lateral edge of an oblique prism an altitude of the prism? Why?

6. In finding the volume of an oblique prism a student accidently used the length of a lateral edge in place of the height of the prism. If he made no other errors, was his answer too large or too small?

9-5 Pyramids

Examine carefully the five Models 13, 14, 15, 16, and 17. These are examples of what are called pyramids. What common property do you observe of these five Models?

A pyramid is a figure obtained by joining the vertices of a polygon to a point not in the plane of the polygon, thus forming triangles. The pyramid consists of the interiors of these triangles, the interior of the original polygon, and the segments which bound these interiors. The interiors are called faces of the pyramid, the segments its edges, and the points where two or more edges intersect vertices. The interior of the original polygon is called the base of the pyramid and the other faces its lateral faces. The point to which the vertices of the polygon are joined we shall call the apex of the pyramid. (Many books call this the vertex of the pyramid, but we have chosen the term apex since we also call the corners of the polygon vertices.) The edges meeting at the apex are called lateral edges. For example, in the figure the base is the interior of quadrilateral ABCD, the lateral faces are the interiors of triangles ABQ, BCQ, CDQ, and DAQ the lateral edges are $\overline{AQ}$, $\overline{BQ}$, $\overline{CQ}$, $\overline{DQ}$, and the apex is Q.



Point out the bases, lateral faces, lateral edges, and apex on each of the Models 14 and 16.

Notice that in Models 13, 15, and 16 the bases are squares, so these are called square pyramids. Similarly Model 14 is a hexagonal pyramid. What kind of pyramid is Model 17?

Although there probably was no argument about the answer to the last question, there might be disagreement over identifying the base. All the faces are triangles, so how do we distinguish which one is the base? The answer of course is that we can't. Any one of the four faces can be considered as the base, so this figure can be looked at as a triangular pyramid in four different ways. (Compare the case of the parallelepiped which could be considered a prism in three ways.) Because it has just four faces this figure is generally called a tetrahedron. (The tetra comes from the Greek word for four.) A tetrahedron with its interior is sometimes called a 3 dimensional simplex which was discussed in detail in Unit 8.

Now look again at the five pyramid models. In each case imagine the segment drawn from the apex perpendicular to the plane of the base. This segment is called the altitude and the length of the altitude is the height of the prism. Compare the heights of Models 13, 14, 15, and 16. Laying a ruler across them may help in estimating heights. Do you find the models have equal heights? Model 17 has four heights, depending which face is taken as base. Take the smallest triangle as base and compare the height with that of the other models. Do all five of these models seem to have the same height?

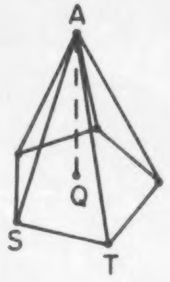
It is not always easy to imagine just where the foot of the altitude will be for a pyramid. In one of the models the altitude coincides with one of the lateral edges, so the foot of the altitude is at a vertex of the base. Find the model and the edge. In another model the foot of the altitude is entirely outside the base. Which model? For the other three the foot of the altitude is somewhere in the base.

The most symmetrical pyramids are called regular pyramids. To be regular, a pyramid must meet two conditions. First, its base must be the interior of a regular polygon. (A regular polygon is one whose sides have equal lengths and whose angles have equal measures.) Which of our models meet this first condition? Second, the foot of the altitude must be at the center of this regular polygon. Which of the models appear to be regular polygons?

It is shown in the Problems below that the second condition is really the same as saying that the lateral edges all have equal lengths, a fact much easier to recognize by looking at the model.

Exercises 9-5

1. Look at the figure which is supposed to show a regular pentagonal pyramid with apex A and altitude $\overline{AQ}$. Since Q is the center of the pentagon it has the same distance from S and from T. Suppose $\overline{AQ}$ is 4 inches long and $\overline{QT}$ and $\overline{QS}$ are each 3 inches long.



(a) How can you find the lengths of $\overline{AS}$ and $\overline{AT}$?

(b) What are these lengths?

(c) Do $\overline{AS}$ and $\overline{AT}$ have equal lengths?

(d) Is triangle AST isosceles?

(e) Can the reasoning be used to show that all five of the lateral edges have the same length?

2. (a) Does the reasoning in the last problem depend on the fact that the base is a pentagon or would it work for any regular polygon?

   (b) Does the reasoning depend on the particular numerical lengths given, or would it apply to any lengths?

   (c) Complete the following statement:

   If a pyramid is regular then its ____________ are all equal.

3. Look again at the figure of Problem 1, with the base a regular pentagon, but this time suppose we know that the lateral edges all have the same lengths but do not know where the foot Q of the altitude is located. To be definite, suppose the height of the prism (i.e. length of $\overline{AQ}$) is 12 inches, and that each of the lateral edges $\overline{AS}$ and $\overline{AT}$ are 13 inches long.

   (a) How can you find the lengths of $\overline{QS}$ and $\overline{QT}$?

   (b) What are these lengths?

   (c) Are they equal?

   (d) Can this reasoning be used to show that the distances from Q to all five vertices of the polygon are equal?

   (e) Does this show Q is the center of the regular polygon?

   (f) Is the pyramid a regular pyramid?

4. (a) Does the reasoning of the Problem above depend on the particular measurements and the fact the base is a pentagon?

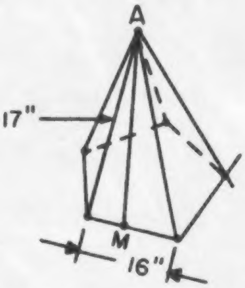
   (b) If not, complete the following statement:

   If, in a pyramid with the interior of a regular polygon as base, the ________________ are all equal in length, then the pyramid is ____________________ .

5. Construct a model of a tetrahedron in which all four faces are equilateral triangles. Such a figure is called a regular tetrahedron.

6. How many altitudes does a regular tetrahedron have?

7. The base of a regular pentagonal pyramid is a regular pentagon, 16.in. on a side. If the lateral edges of the pyramid are each 17 in., find the lateral area of the prism, that is the sum of the areas of all five lateral faces. Hint: Draw segment $\overline{AM}$ from Apex A to a midpoint of a side of the pentagon. This is the altitude of this triangular face and is called the slant height of the regular pyramid.



8. A regular square pyramid has a base which is the interior of a square 10 inches on a side. Its slant height (see problem above) is 12 inches.

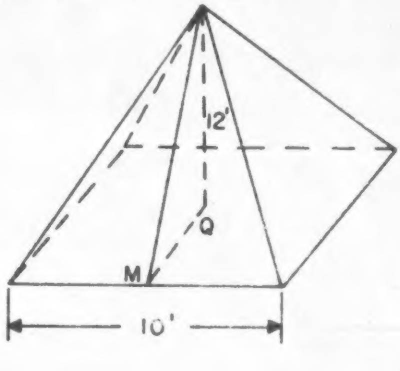
   (a) Find its total area (sum of areas of lateral faces and the base).

   (b) Find the lengths of the lateral edges.

9. The base of a square pyramid is the interior of a square 10 feet on a side. The altitude of the pyramid is 12 feet.

   (a) Find the total area.

   (b) Find the lengths of the lateral edges. Hint: How far is it from Q to M ? Use this to find the slant height.



9-6 Volumes of Pyramids

Can we do anything now about finding volumes of the interiors of pyramids?

In the section on prisms we found it useful to consider models made up of stacks of cards. Perhaps you and your classmates would like to make a similar model for a pyramid. If so, get some heavy cardboard (such as grocery cartons) and make a series of squares to pile on each other. As a suggestion make the bottom one 6 inches on a side, the next one $5\frac{7}{8}$ in a side, etc., going down by $\frac{1}{8}$ inch each time. Theoretically you will have 48 squares, but actually you will have to omit the very top ones as they get too small to work with. However, you should be able to go up at least to the 1 inch by inch square. To avoid having the squares fall apart when you move them, make a hole in the center of each one and run a cord through them, preferably an elastic cord to hold them firmly together.

If you want a larger model, start with a square one foot on a side. This will take twice as many layers and eight times as much cardboard. A deluxe model might be made by cutting the squares out of a $\frac{1}{8}$ inch masonite or something similar in your wood shop. The larger model would take a little over 32 square feet of material, the smaller a little more than 4 square feet.

Such a model should convincingly remind you of a square pyramid, though of course there are irregularities at the edges as in the case of the prism. By shoving the squares around you can make this model assume approximate shapes of all kinds of square pyramids. When the squares are piled up with the center holes vertically above each other it appears as a regular pyramid like our Model 13. By pushing it to one side you move the apex so it is no longer above the center of the base. You can very probably push it far enough so the apex is over a corner of the base as in our Model 15, and possibly even into the position of Model 16 where the perpendicular from the apex is outside the base.

In all this moving around we clearly have not changed the base of the pyramid or its height, which is after all just the thickness of our stack of squares. Moreover, we have not changed the amount of cardboard in the pile. It looks like a good guess then that any two pyramids with congruent bases and equal heights have equal volumes.

Let us try this out on Models 13, 15, and 16 which clearly have congruent square bases and whose heights are the same as we saw earlier. Fill Model 13 with salt and try emptying it into 15 and then into 16. Do your results confirm the guess above?

On the basis of this experiment and the evidence of our cardboard model we write the following property:

Property 20. If two pyramids have congruent bases and equal heights they have equal volumes.

To find what the actual volume of a pyramid is however, we must eventually compare it with some figure whose volume we know. As an experiment take Model 13, the regular square pyramid and Model 4, the rectangular right prism. How do the bases of these two models compare (if we take the small square as the base for Model 4)? How do their heights compare? Do you agree they have congruent bases and equal heights? The interior of Model 4 is clearly larger than the interior of Model 13, but how much larger? Fill Model 13 with salt and pour it into Model 4. Keep on doing this until Model 4 is full. According to your results the interior of Model 4 is how many times that of Model 13?

Repeat the experiment with Model 14 and Model 5. Did you get the same multiple in this case? Make a third trial with Model 17 and Model 6. On the basis of these experiments do you agree with the following property?

Property 21. The volume of a pyramid is one third that of a prism whose base is congruent to the base of the prism and whose height is the same as that of the prism.

Since by Property 19 we know how to find the volume of a prism this leads at once to a rule for finding the volume of any pyramid.

Property 22. The number V of cubic units of volume in a pyramid is given by the formula:

$$V = \frac{1}{3}Bh$$

where B stands for the number of square units of area in the base and h the number of linear units in the height.

Exercises 9-6

1. Model 13 has a square base of $1\frac{1}{2}$ inches on a side and a height of 4 inches. Check these measurements with your model. Then find the volume of the interior of Model 13.
2. Check the result of the last problem by taking Model 1 and finding how many times it must be emptied into Model 13 to fill it.
3. Model 17 has a base which is a right triangle. Measure the legs and find its area. Since the height of the prism is 4 inches,

find the volume of its interior. Do you get the same result as for Problem 1?

4. Find the volume of Model 14. Notice the base is a regular hexagon whose sides are each 1 inch. How will you find the area of the interior of this hexagon?

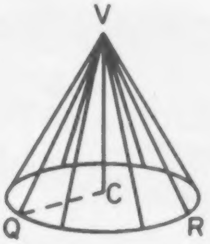
5. Make a rough check of the results of Problem 4 by using Model 1 to see how many cubic inches Model 14 really holds. This will not come out exactly but you should be able to get the volume within half a cubic inch.

6. A pyramid has a height of 12 feet and a volume of 324 cu. ft. What is the area of its base? If it is a square pyramid, what is the length of each side of the base?

7. Make Model 18.

9-7 Cones

Anyone who has eaten an ice cream cone has at least a rough idea of the figure called a cone, or more strictly a right circular cone. Let a circle be drawn as shown below, with center C, and let V be a point not in the plane of the circle so that segment $\overline{VC}$ is perpendicular to this plane.

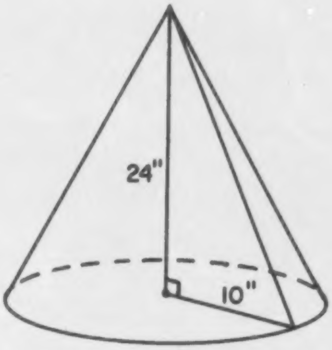


Draw all the segments from V to the points of the circle. The union of all these segments, together with the interior of the circle, forms a right circular cone. The interior of the circle is called the base of the cone, and the union of the segments is its lateral surface. The point V is called the vertex of the cone. In the description right circular cone the word circular indicates that the base is the interior of a circle and the word right means that $\overline{VC}$ is perpendicular to the plane of the circle. Here we consider only right circular cones and when the word cone is used it will mean this type.

Segment $\overline{VC}$ is called the altitude of the cone, and the length of this segment is the height of the cone. If Q is a point of the circle, what kind of triangle is VCQ ? Why? If you know the height of the cone and the radius of its base can you find the length of $\overline{VQ}$? How? If R is another point of the circle, do $\overline{VQ}$ and $\overline{VR}$ have the same length? This constant distance from vertex V to the different points of the circle is called the slant height of the cone.

If h is the number of linear units in the height of the cone, r the number of linear units in the radius, and s the number of linear units in the slant height, write an equation connecting h, r, and s . If you know any two of these numbers can you find the third one from this equation?

As an example, suppose the radius of the base of a cone is 10 in. and the height is 24 in. What is the slant height of the cone?
Did you find the slant height to be 26 in.?



Examine Model 18. Point out the base, the vertex, and the lateral surface. Approximately what is the slant height? Do you find it is about $4\frac{1}{8}$ inches? Do you find the radius a little less than an inch? Writing these as decimals and rounding to one decimal place, we may take the slant height as 4.1 inches and the radius as 0.9 inches. What is the height of the model?

How can we find the volume of a cone? Suppose we use the method used on pyramids and compare a cone with a cylinder having the same height and same sized base. Take Models 18 and 8. Compare their bases. Are the circles the same size? Do the two models appear to have equal heights? How did you test this?

Now fill Model 18 with salt and empty it into Model 8. Continue till Model 8 is full. On the basis of this experiment, the volume of Model 8 is how many times that of Model 18? This illustrates the following property.

Property 23. The volume of the interior of a cone is one third that of a cylinder of the same height and whose base has the same radius.

Since we have already learned to find the volume of a cylinder, this leads at once to a rule for finding the volume of a cone.

Property 24. If r is the number of linear units in the radius of the base of a cone and h the number of linear units in the height, the number V of cubic units in the volume of its interior is given by the formula

$$V = \frac{1}{3} \pi r^2 h.$$

Since πr^2 is actually the number of square units B of area in the base, the formula could be written as

$$V = \frac{1}{3} Bh.$$

Comparing this with Property 22 shows that we have the same rule for finding the volume of a cone as for a pyramid.

As an example refer back to the cone mentioned above where the radius of the base was 10 inches and the height 24 inches. Then $r = 10$, $h = 24$, so by the formula above

$$V = \frac{1}{3} \pi (10)^2 24 = 800\pi$$

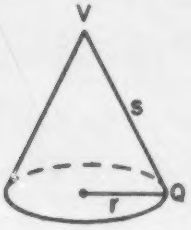
and the volume is 800π cu. in. or about 2512 cu. in.

Exercises 9-7

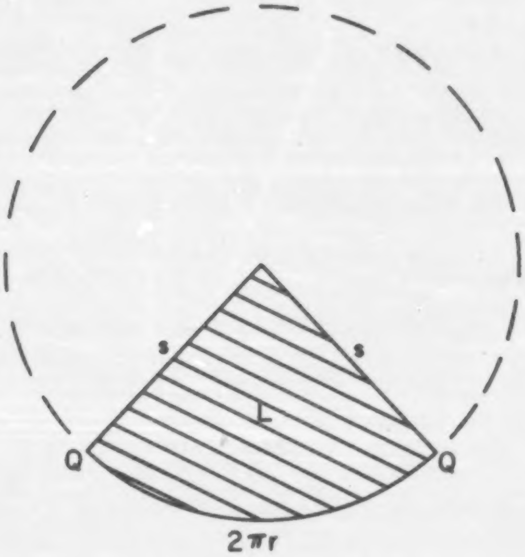
*Lateral Area of a Cone

To see how to find the lateral area of a cone, look at Model 18. If we take it apart again, the lateral surface goes back into a sector of a circle as shown in the pattern for the model.

(Notice a sector of a circle is bounded by two radii and a part of the circle.) That is the model which looks like this



flattens out into a sector of a circle that looks like this.



The lateral area of the cone has the same measure as the area of the shaded part we are trying to find. The two points marked Q in the figure come from the same point of the model. The rest of the large circle is shown in dotted lines to help follow the reasoning.

Let s be the number of units in the slant height of the cone and r be the number of units in the radius of its base. Do the markings on the figure above on the two segments and the arc show the correct number of units in their lengths? Why?

Now in a sector of a circle, such as we have above, the area is proportional to the arc. For example, if the arc between the two points marked Q is one quarter of the circle, then the shaded area is one quarter of the interior of the circle. But the circumference of the circle is $2\pi s$ and its area is πs^2. If L stands for the number of square units in the shaded area we find, therefore.

$$\frac{2\pi r}{2\pi s} = \frac{L}{\pi s^2}$$

How do you solve the equation for L ? What value do you find for L ?

This reasoning justifies the following conclusion:

Property 25. If the slant height of a right circular cone is s units and the radius of its base r units, the number L of square units in its lateral area is given by the formula:

$$L = \pi r s .$$

As an example refer again to the cone where the radius of the base is 10 inches long and the height 24 inches. You recall we found the slant height is 26 inches. In this problem we have, therefore $r = 10$; $s = 26$, so

$$L = \pi\, 10 \cdot 26 = 260\, \pi \approx 816.4$$

and the lateral area is about 816.3 square inches.

Exercises 9-7

1. If T stands for the number of square units in the total area of the cone (counting the base) write a formula for T in terms of r and s .

2. The slant height of a cone is 12 ft. and the radius of its base 3 ft. Find its lateral area and its total area.

3. A cone has a height of 12 ft. and its slant height is 15 ft. Find the radius, the lateral area, the total area, and the volume.

4. The radius of the base of a cone is 2 ft. long and the volume of its interior is 4π cubic inches. Find its height, slant height, and lateral area.

5. Construct Models 19a, 19b, and 19c. Actually 19a and 19b are identical except for the lettering and can be cut out at the same time. Be sure to put the letters on however as we will need them to identify the different vertices. Notice that the letters do not refer to particular angles but identify a particular vertex after the model is assembled.

* 9-8 Disection of a Prism

According to our experiments with pyramids, the volume of the interior of a pyramid is one third that of a prism having the same height as the pyramid and having a base which is congruent to the base of the pyramid. It is natural to ask whether we could see this by putting together three identical pyramids to form the prism. Unfortunately, a little experimentation seems to show this is not possible. However, we can get a kind of substitute, as we shall see.

Examine Models 19a, 19b, and 19c. They are all tetrahedrons or triangular prisms. First compare Models 19a and 19b. How does face ABC of Model 19a compare with face SRQ of Model 19b? How do their heights compare if we consider these faces as bases? (Actually these questions are a little ridiculous since we have already noticed the patterns are identical for the two models so all their measurements must agree.) In any case the two tetrahedrons ABCQ and QRSC , that is Models 19a and 19b have interiors with equal volumes.

Now compare Models 19a and 19c. This time the models definitely do not look alike. However, compare face ABQ of Model 19a with face BCR of 19c. Do you find them congruent? Place the models on the desk with these faces in contact with the top of the desk. Notice that in these positions you can push the models together so that the two faces marked BCQ coincide. What can you say of the heights of these two models when placed in this position? Models 19a and 19c, when looked at in this way, are triangular pyramids with congruent bases and equal altitudes.

What can you say about the volumes of their interiors? What Property are you using?

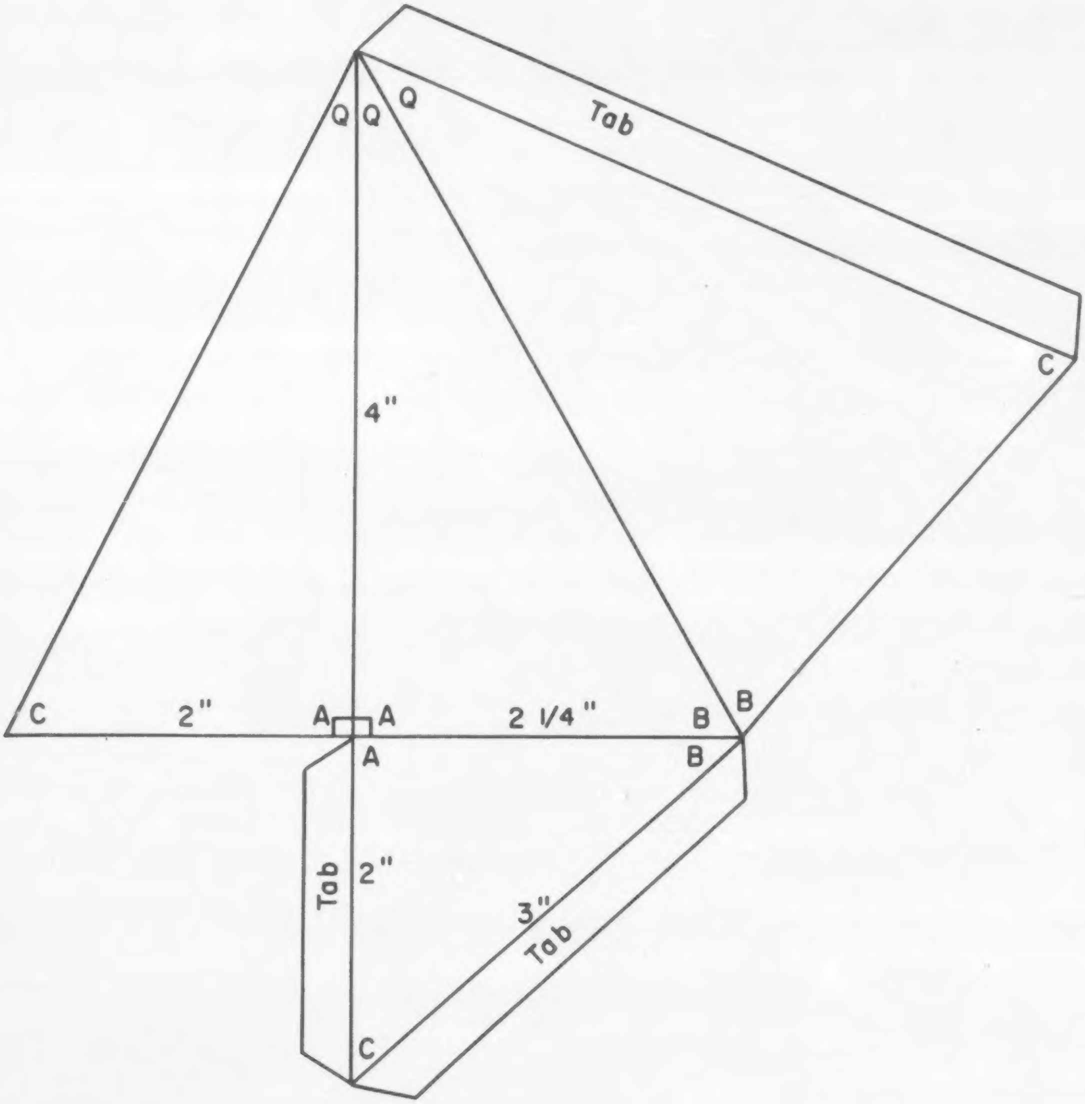
You should have concluded that the three Models 19a, 19b, 19c have interiors with equal volumes. Now put the three models together so that faces BCQ of Models 19a and 19c coincide and so that faces QRC of 19b and 19c coincide. What is the resulting figure? Is it a triangular prism?

These three models with equal volumes can thus be assembled to form a prism whose base is the same as the face ABC of Model 19a, and whose height is the same as that of 19a. This shows again the result stated in Property 21. Actually the work is just the process of disecting a prism into tetrahedrons which you discussed in Unit 8 except that here we have been particularly interested in the volumes of the pieces.

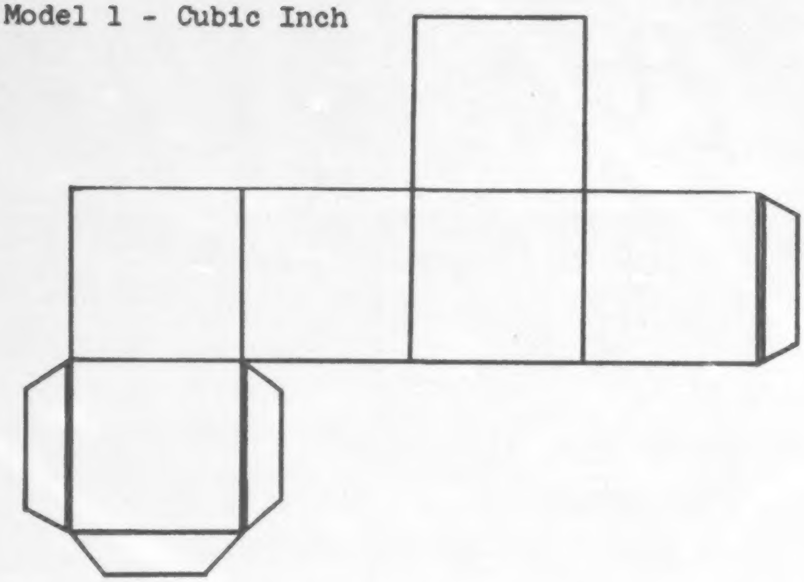
If we imagine Model 19a as originally given, we can think of Models 19b and 19c as two more tetrahedrons which have been invented having the same volume as 19a and so that they can be combined with 19a to produce a prism of the same base and height. In this particular case the base of 19a is an equilateral triangle, and one of the lateral edges is perpendicular to the plane of the base. Could this still have been done if ABCQ were any triangular prism? The answer is actually yes. In the Exercise 9-8 you are asked to produce the other two models for a more general tetrahedron.

Exercises 9-8.

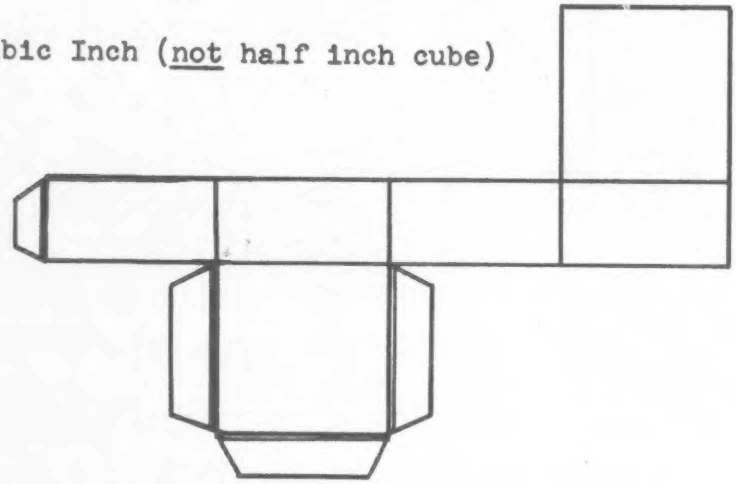
BRAINBUSTER. Construct the model for which the pattern is given below. Letter the vertices as shown and think of it as a triangular pyramid with base the interior of triangle ABC. Design and construct models of two other tetrahedra having the same volume as this given one and which, combined with it, produce a prism having the same base and height.



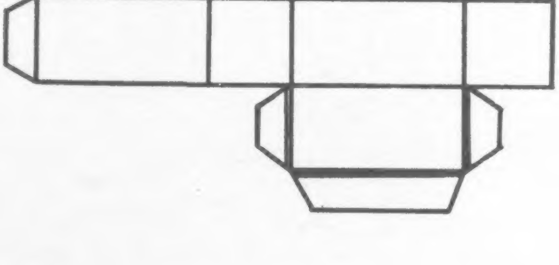
Model 1 - Cubic Inch



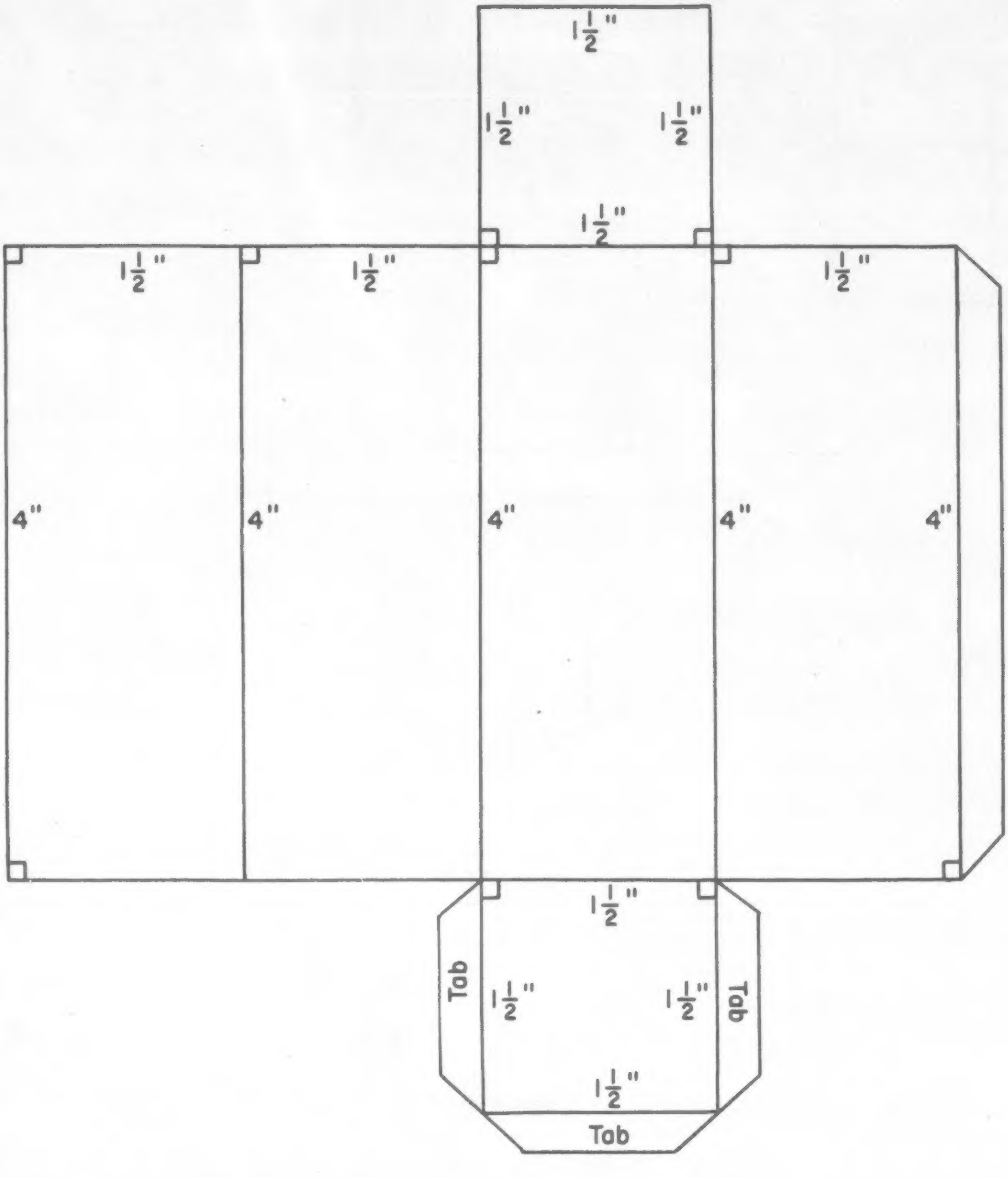
Model 2 - Half Cubic Inch (not half inch cube)



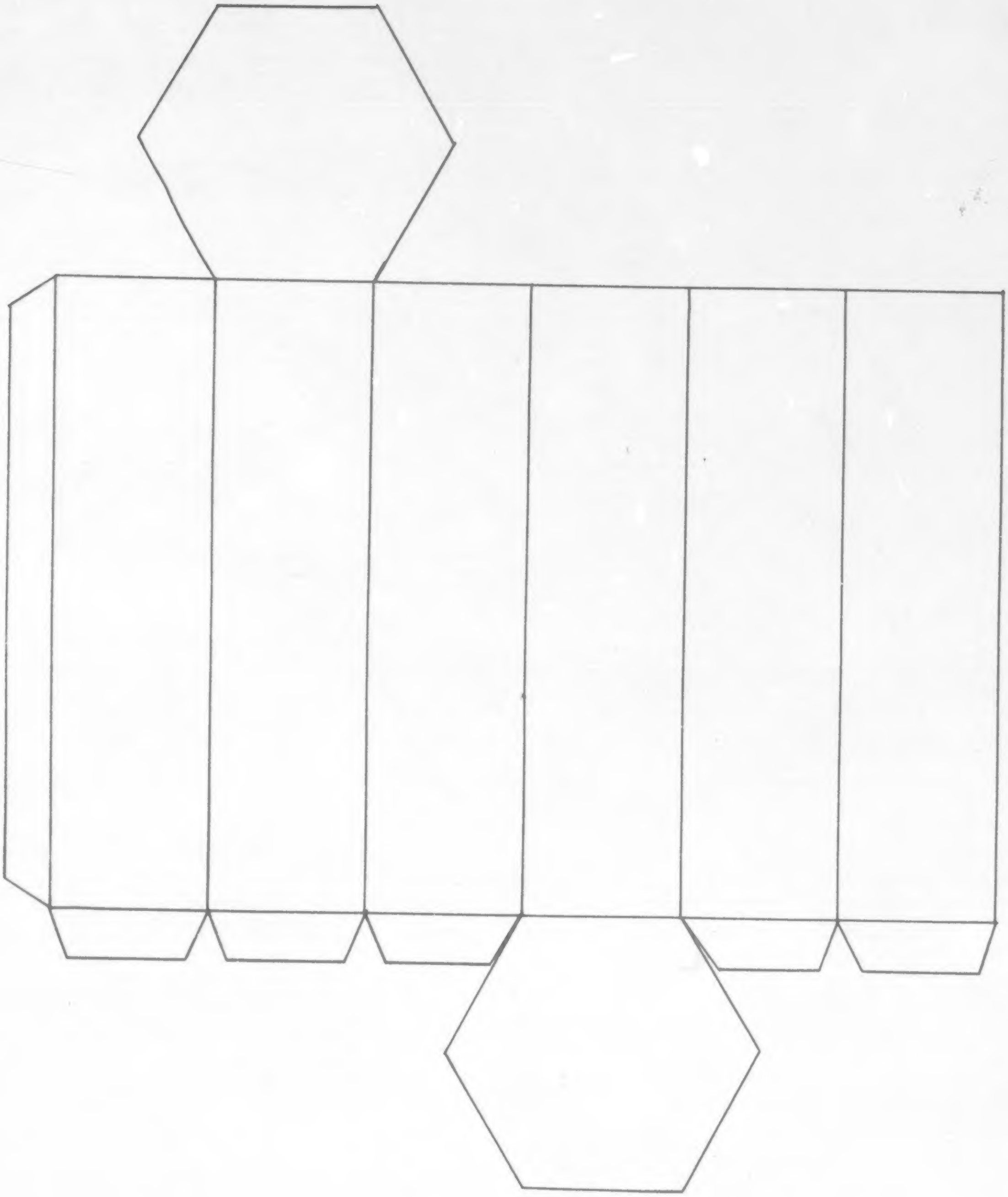
Model 3 - Quarter Cubic Inch (not quarter inch cube)



Model 4 - Rectangular Right Prism


1½"
1½"
1½"
1½"
1½"
1½"
1½"
4"
4"
4"
4"
4"
1½"
Tab
1½"
1½"
Tab
1½"
Tab

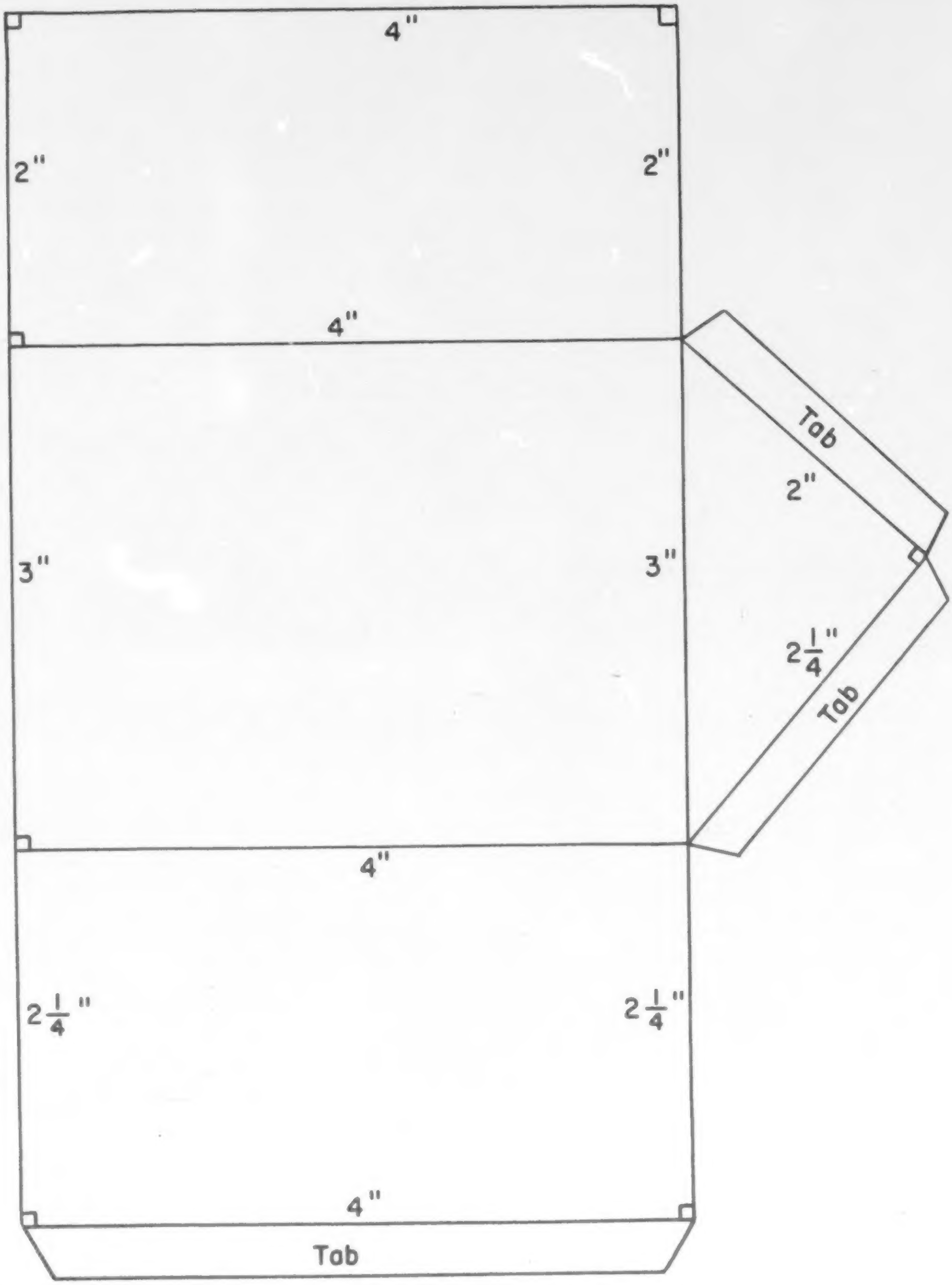
Model 5 - Right Hexagonal Prism



Model 6 - Right Triangular Prism.

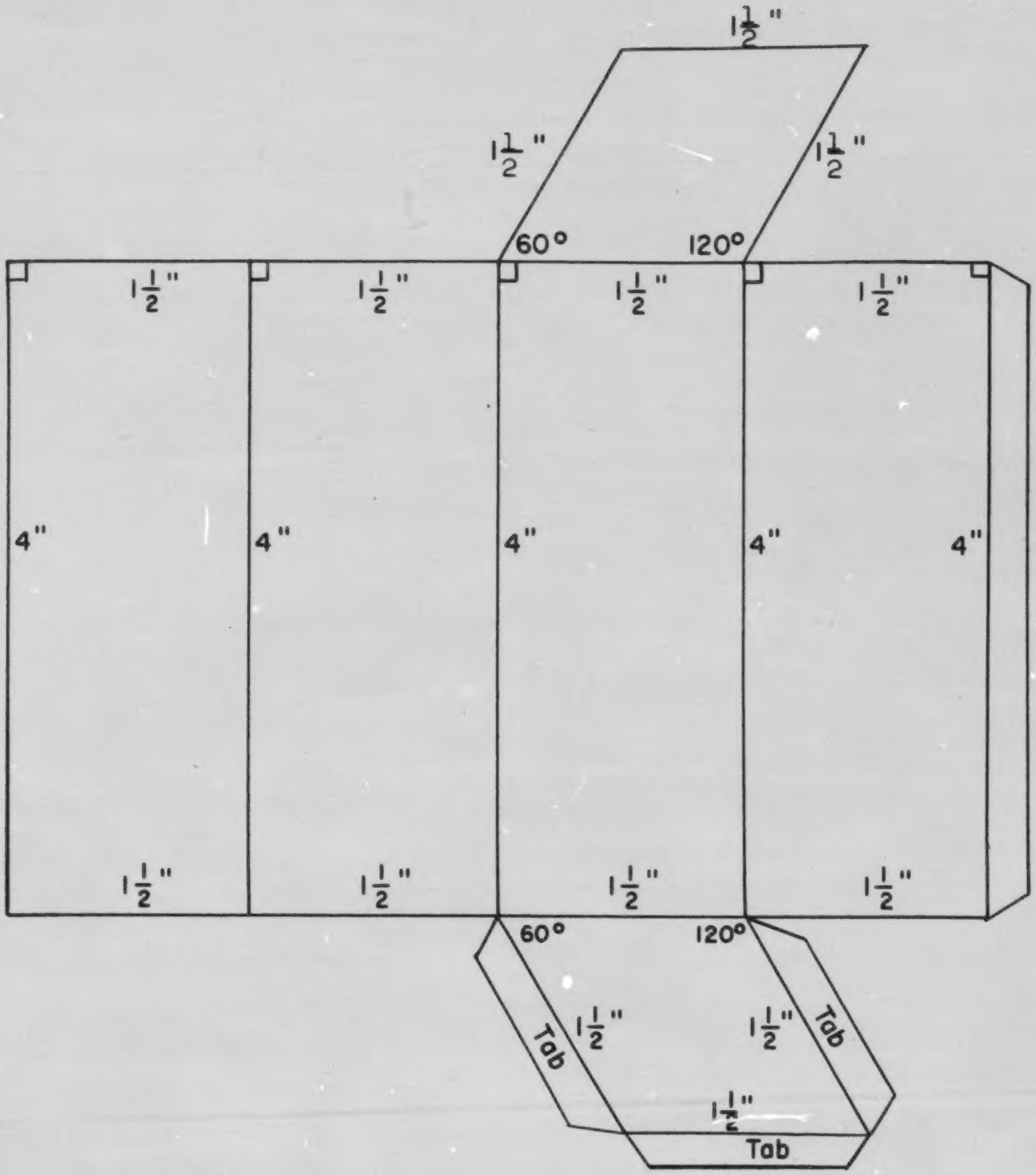
(Base is Interior of a Right Triangle)

Make an extra copy of the triangle to use for the other base, but use only one tab so the top can be opened.


4"
2"
2"
4"
Tab
2"
3"
3"
$2\frac{1}{4}$"
Tab
4"
$2\frac{1}{4}$"
$2\frac{1}{4}$"
4"
Tab

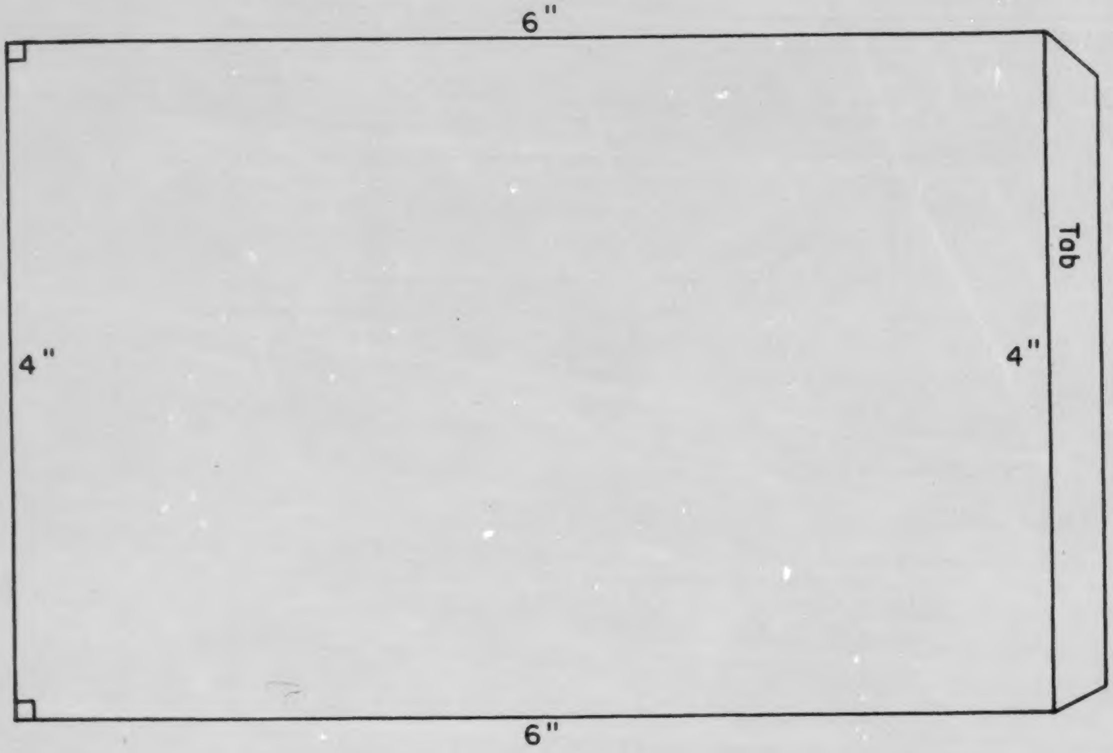
Model 7 - Right Prism with Rhombus as Base

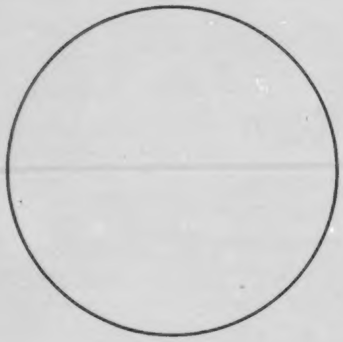
(also Parallelopiped)


1½"
1½"
1½"
60°
120°
1½"
1½"
1½"
1½"
4"
4"
4"
4"
4"
1½"
1½"
1½"
1½"
60°
120°
1½"
1½"
Tab
Tab
1½"
Tab

Model 8 - Right Circular Cylinder

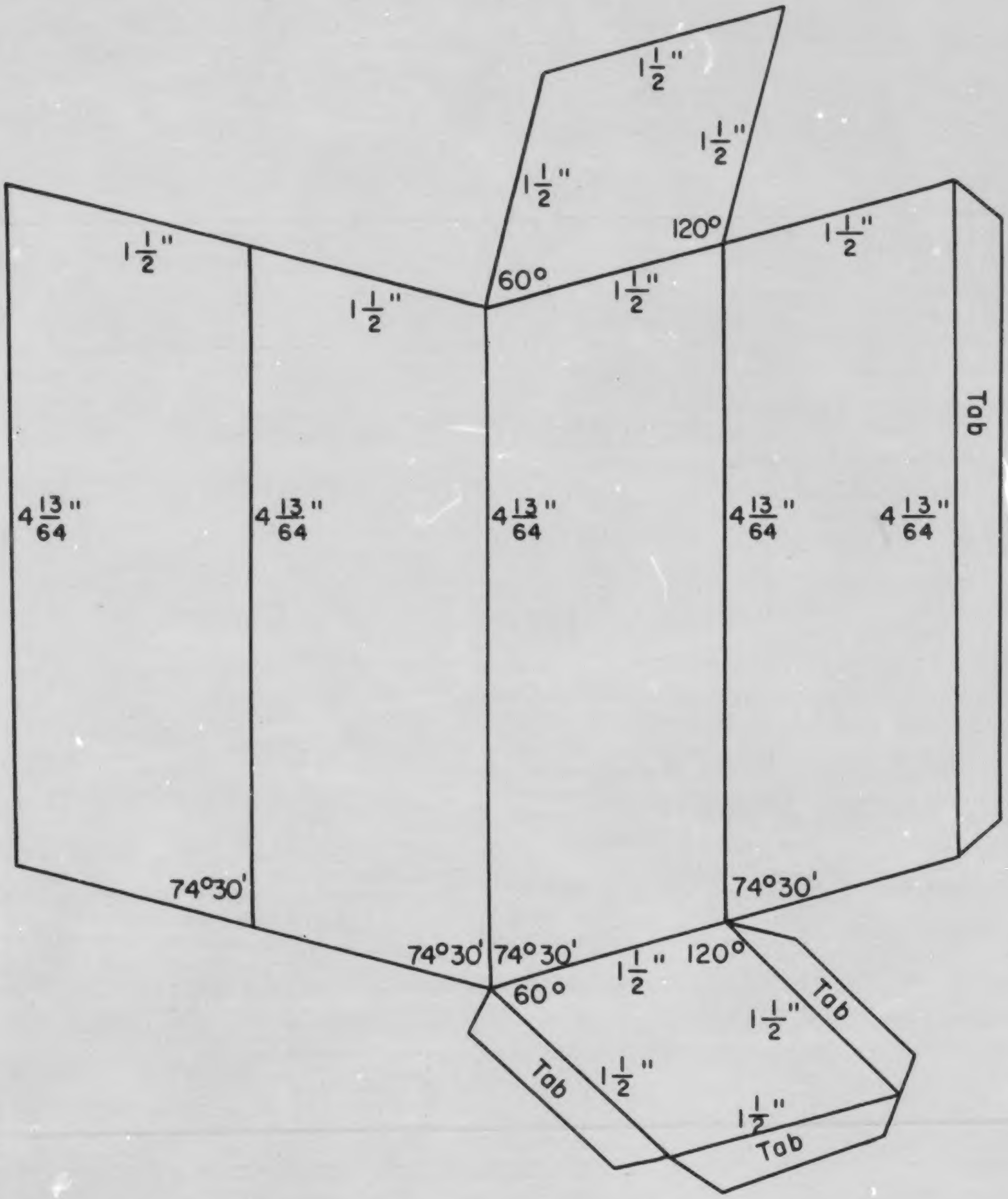
It will be easier to draw the circular bases with your own compass using the radius of the circle below rather than trying to trace the circle as shown. Make two copies of the circle for the two bases. Attach the lower base firmly (with scotch tape) but attach the top base only at one point so it can be readily opened.





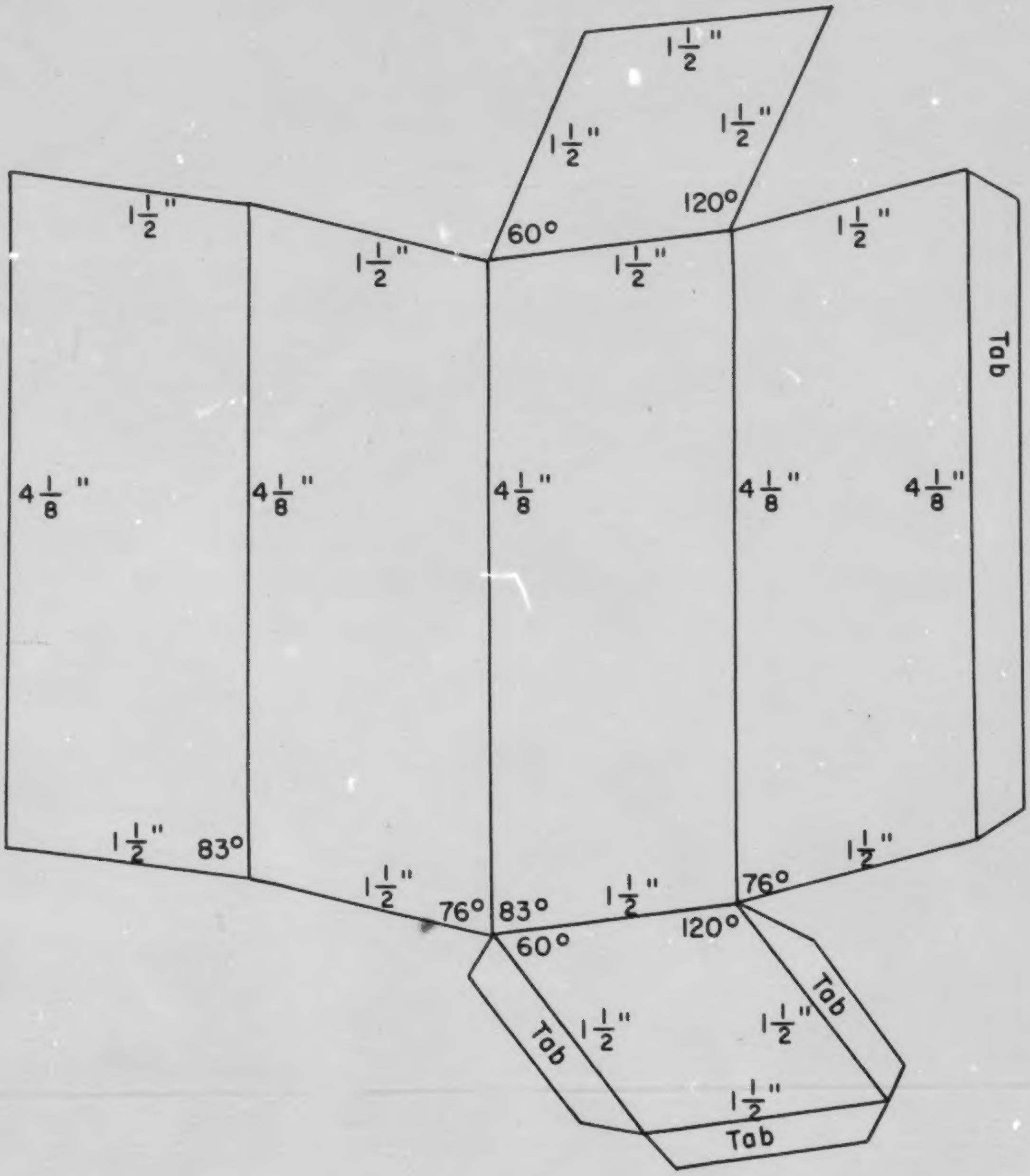
Model 9 - Oblique Prism with Rhombus as Base

(Also Parallelepiped)


$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
120°
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
60°
$1\frac{1}{2}$"
$1\frac{1}{2}$"
Tab
$4\frac{13}{64}$"
$4\frac{13}{64}$"
$4\frac{13}{64}$"
$4\frac{13}{64}$"
$4\frac{13}{64}$"
74°30'
74°30'
74°30'
74°30'
$1\frac{1}{2}$"
120°
60°
Tab
$1\frac{1}{2}$"
Tab
$1\frac{1}{2}$"
$1\frac{1}{2}$"
Tab

Model 10 - Oblique Prism with Rhombus as Base

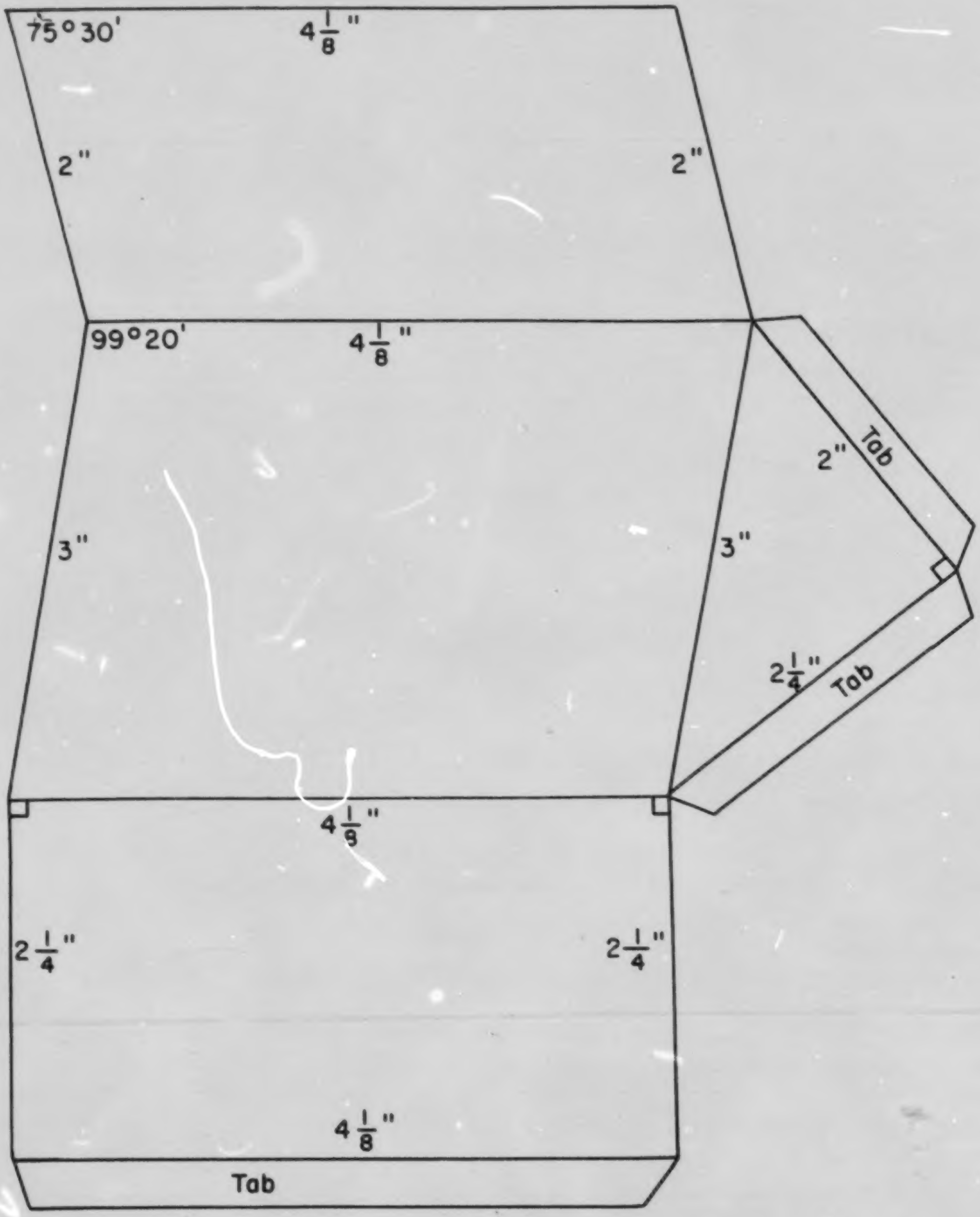
(also Parallelepiped)


1 1/2"
1 1/2"
1 1/2"
120°
60°
1 1/2"
1 1/2"
1 1/2"
1 1/2"
Tab
4 1/8"
4 1/8"
4 1/8"
4 1/8"
4 1/8"
1 1/2"
83°
1 1/2"
76°
83°
1 1/2"
76°
1 1/2"
60°
120°
Tab
Tab
1 1/2"
1 1/2"
1 1/2"
Tab

Model 11 - Oblique Triangular Prism

(Base is Interior of a Right Triangle)

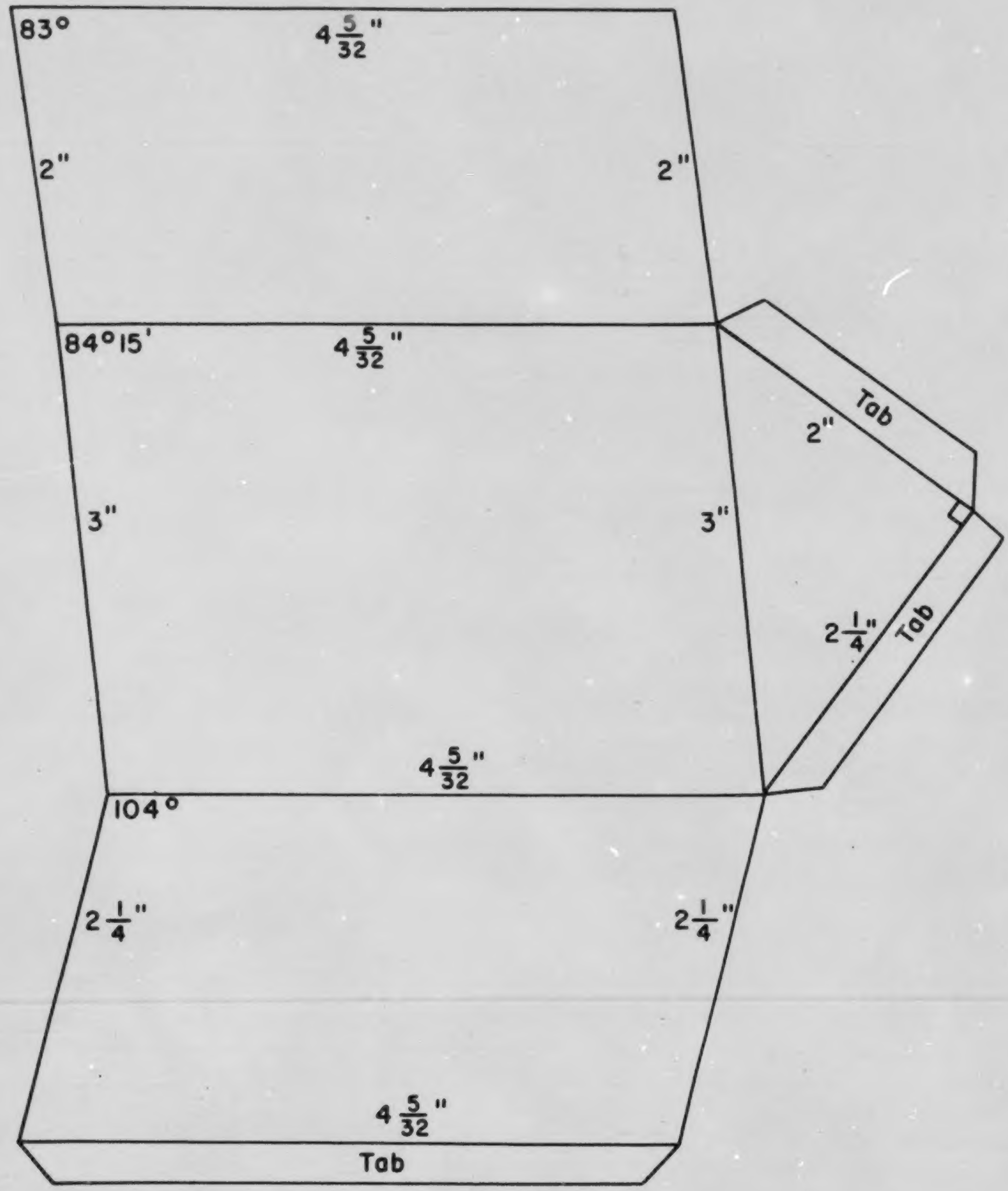
Make an extra copy of the triangle to use for the other base, but use only one tab in attaching it, so the top can be opened if desired.


75°30'
4 1/8"
2"
2"
99°20'
4 1/8"
Tab
2"
3"
3"
2 1/4"
Tab
4 1/8"
2 1/4"
2 1/4"
4 1/8"
Tab

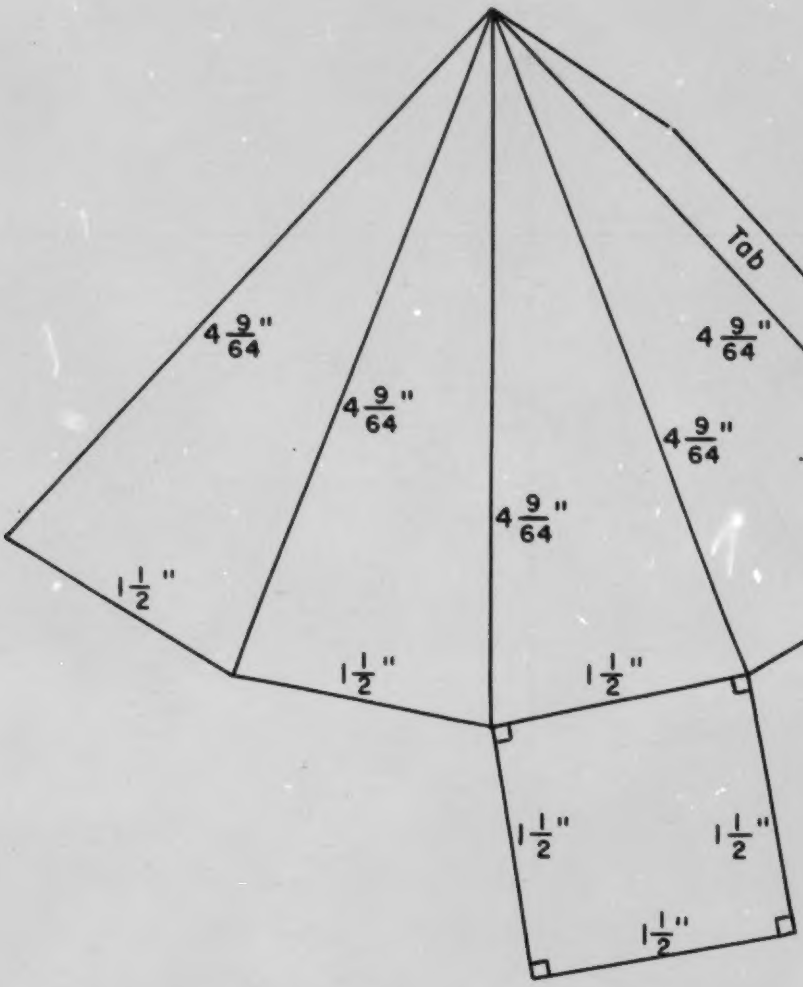
Model 12 - Oblique Triangular Prism

(Base is the Interior of a Right Triangle)

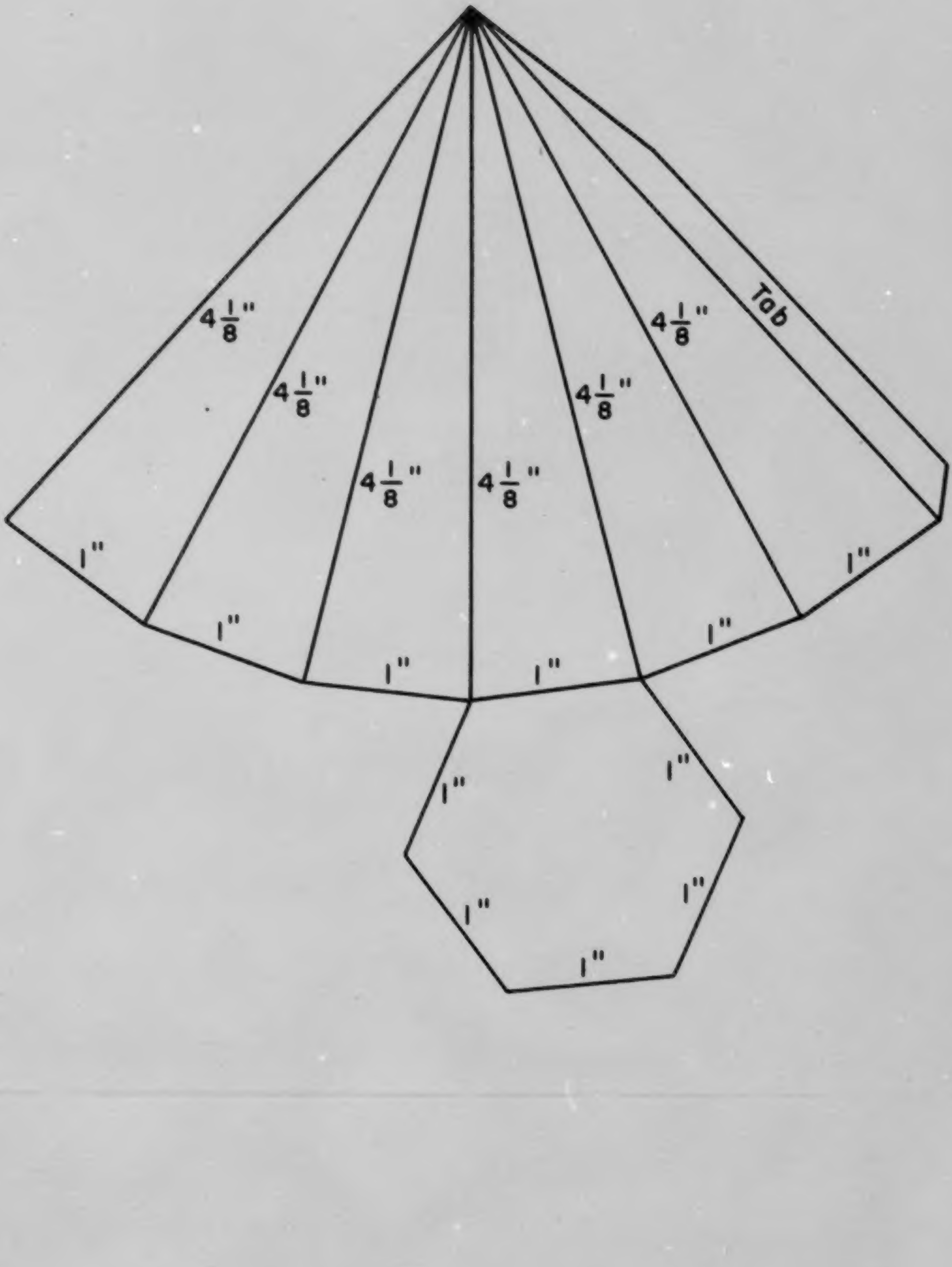
Make an extra copy of the triangle to use for the other base, but use only one tab in attaching it so the top can be opened if desired.


83°
4 5/32"
2"
2"
84°15'
4 5/32"
Tab
2"
3"
3"
2 1/4"
Tab
4 5/32"
104°
2 1/4"
2 1/4"
4 5/32"
Tab

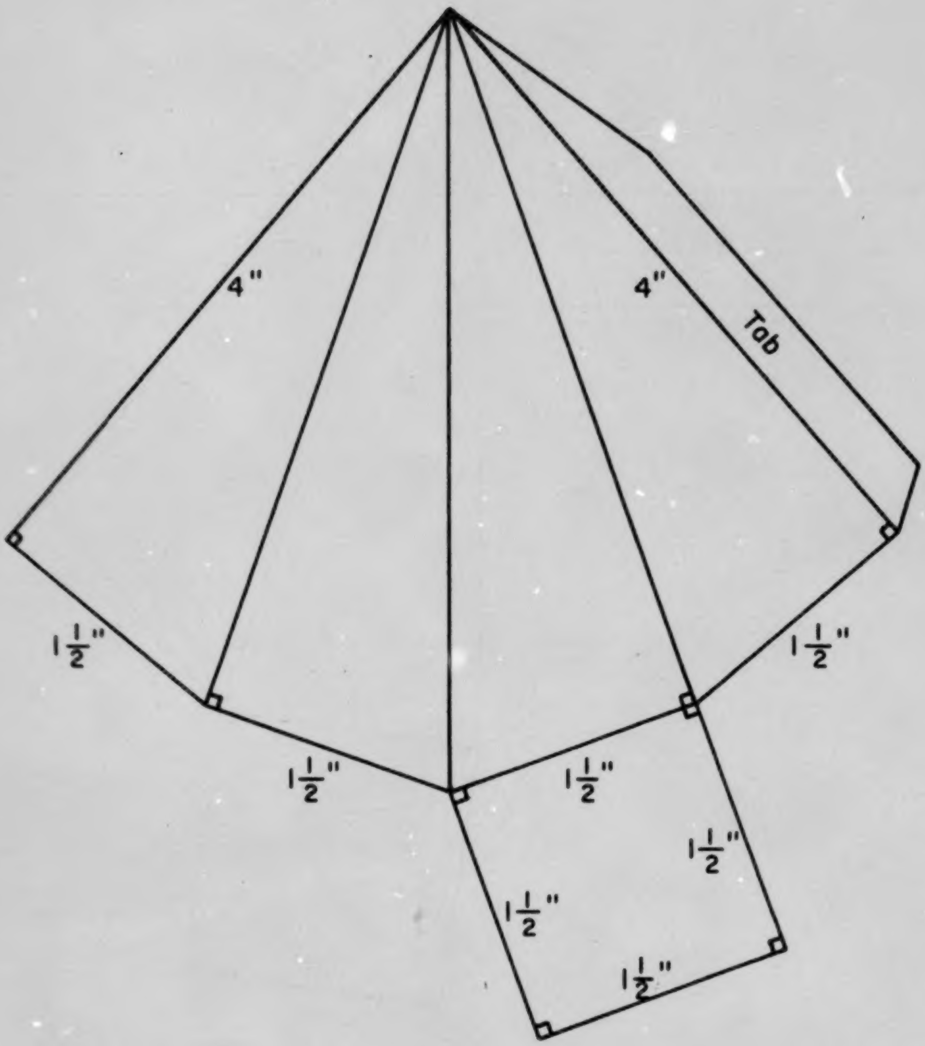
Model 13 - Regular Square Pyramid


Tab
4 9/64"
4 9/64"
4 9/64"
4 9/64"
4 9/64"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"

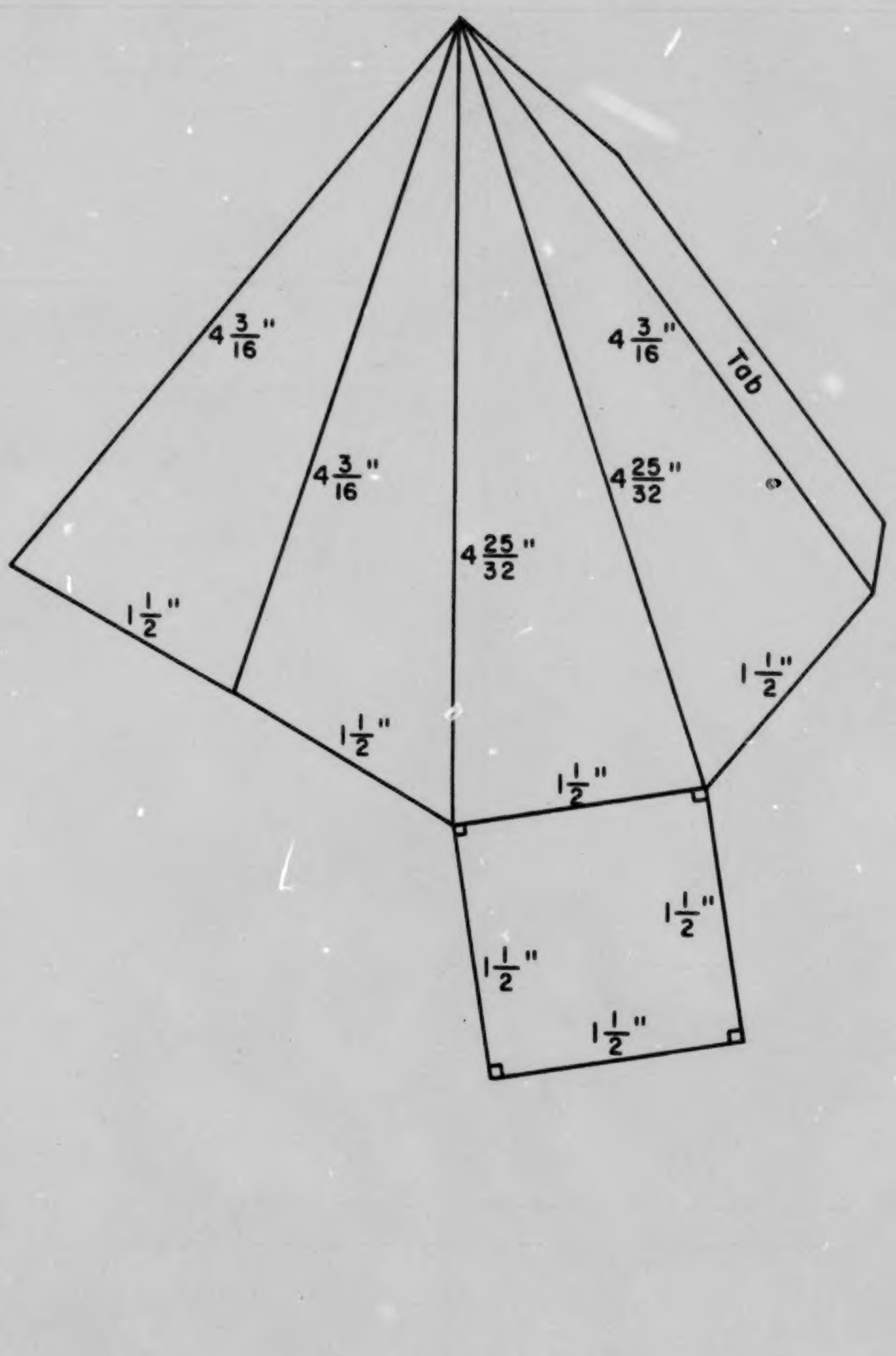
Model 14 - Regular Hexagonal Pyramid


4 1/8"
4 1/8"
4 1/8"
4 1/8"
4 1/8"
4 1/8"
Tab
1"
1"
1"
1"
1"
1"
1"
1"
1"
1"
1"
1"

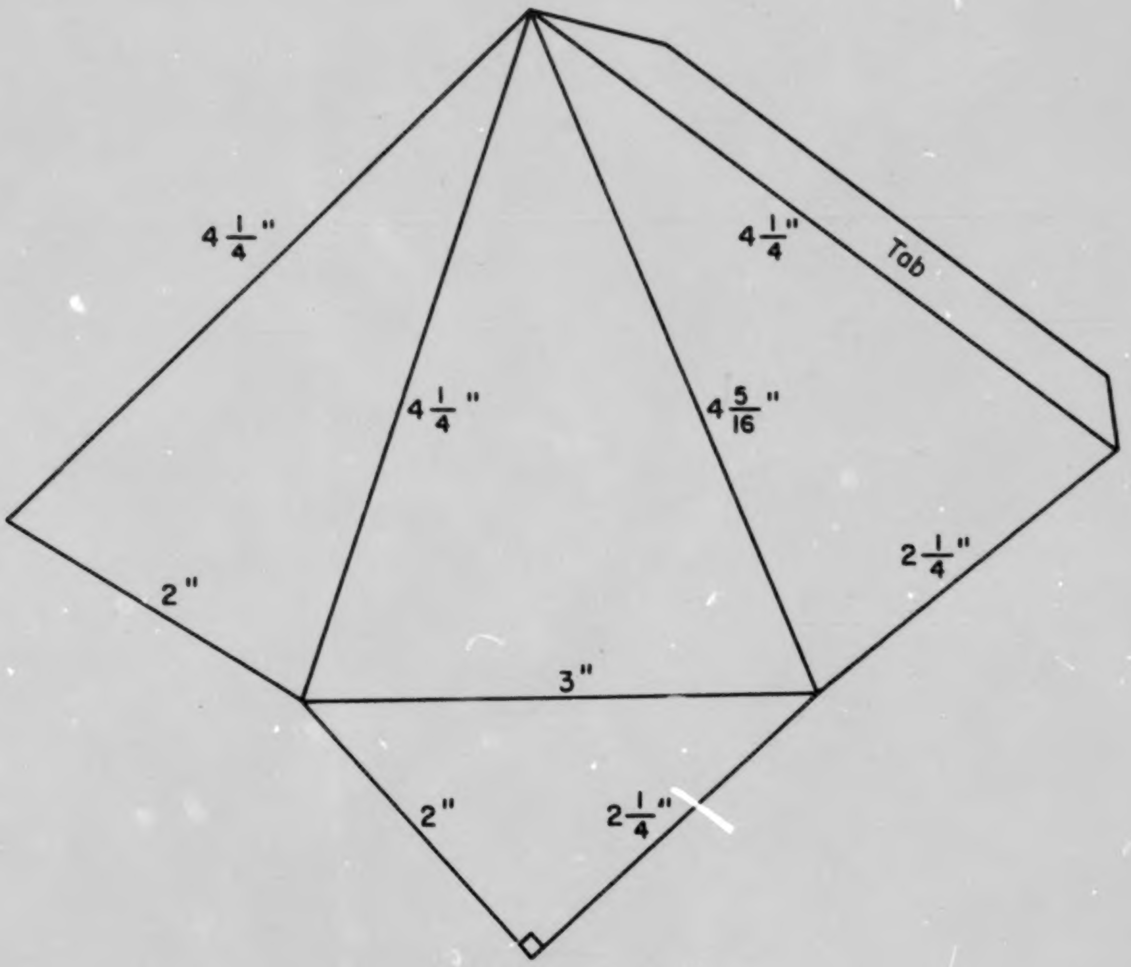
Model 15 - Square Pyramid


4"
4"
Tab
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"

Model 16 - Square Pyramid

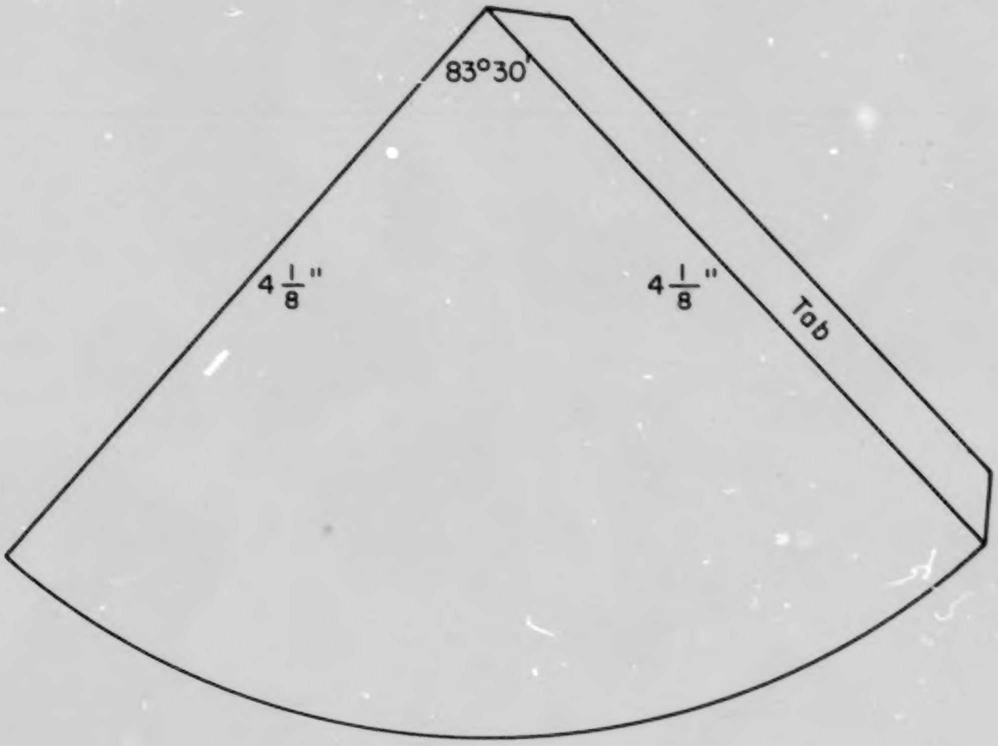

4 3/16"
4 3/16"
4 25/32"
4 25/32"
4 3/16"
Tab
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"

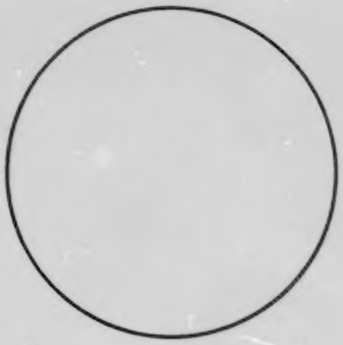
Model 17 - Triangular Pyramid (Tetrahedron)


4 1/4"
4 1/4"
Tab
4 1/4"
4 5/16"
2"
2 1/4"
3"
2"
2 1/4"

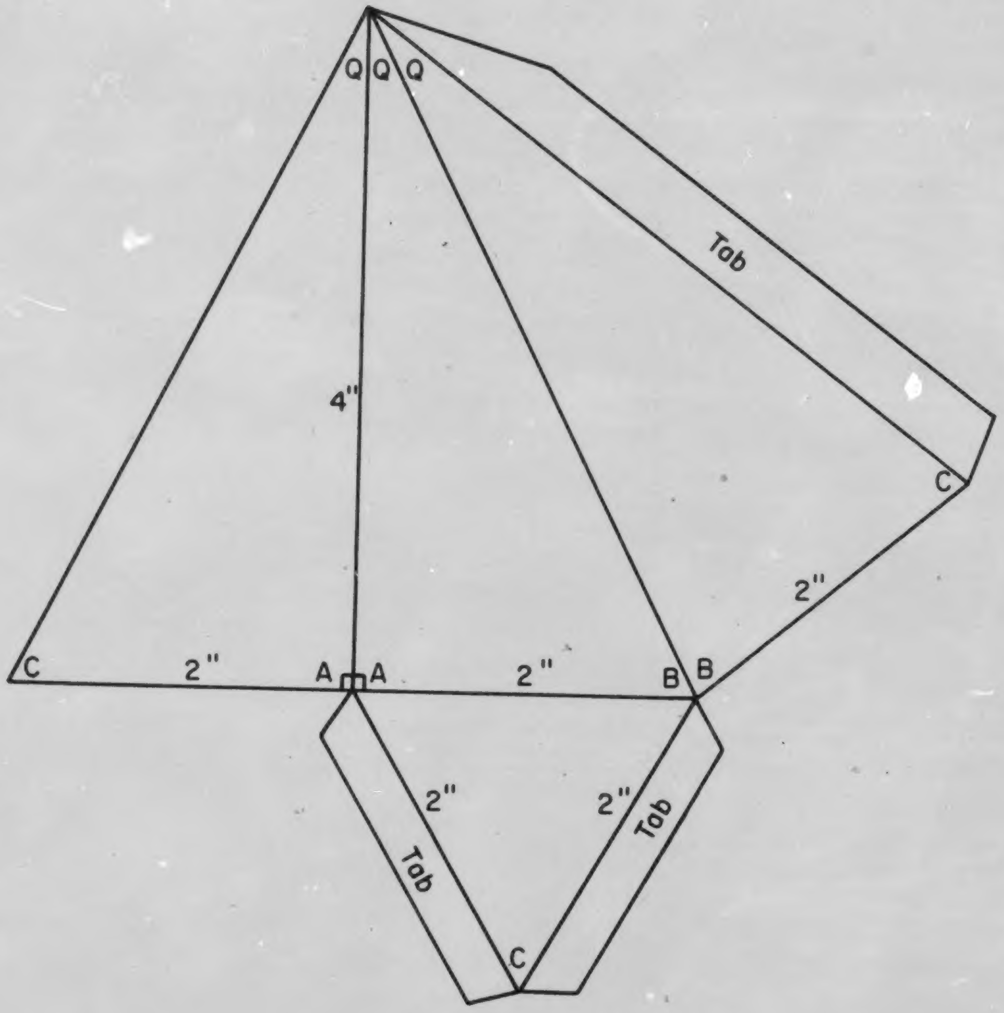
Model 18 - Right Circular Cone

It will be better to draw these circles with your own compass using the radii of the circles drawn rather than trying to trace them. The radius of the small circle is supposed to be the same as in Model 8.

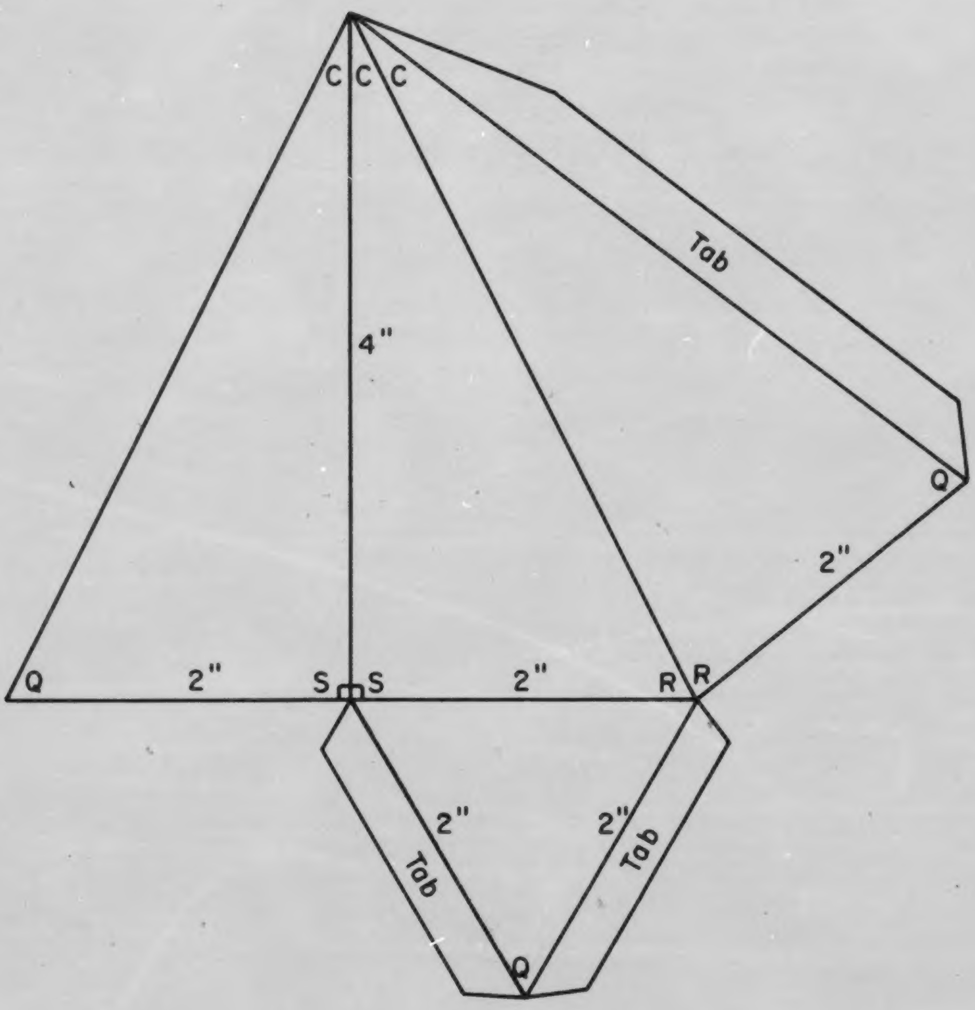




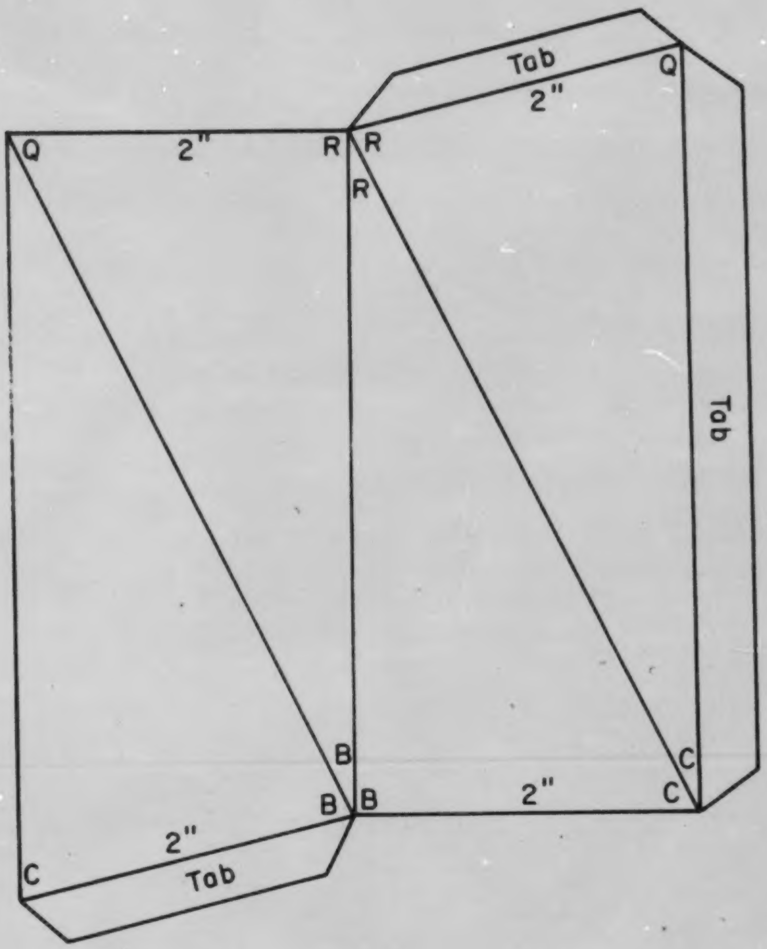
Model 19a


Q
Q
Q
Tab
4"
C
2"
C
2"
A
A
2"
B
B
2"
2"
Tab
Tab
C

Model 19b


C
C
C
4"
Tab
Q
2"
Q
2"
S
S
2"
R
R
2"
2"
Tab
Tab
Q

Model 19c


Tab
Q
2"
Q
2"
R
R
R
Tab
B
C
B
B
2"
C
C
2"
Tab

The remaining segments not labeled for length are the same length as the segments in Models 19a and 19b joining the same end points. That is, segment $\overline{BQ}$ here has the same length as segment $\overline{BQ}$ in Model 19a.

Summary of Properties for Unit 9

Property 1. The number A of square units of area in the interior of a rectangle is the product of the numbers of linear units ℓ and w in the length and width respectively. Thus $A = \ell w$.

Property 2. The number of square units A in the area of the interior of a square whose edge is s units in length is the product of the numbers of linear units in its length(s) and its width(s). Thus $A = s \cdot s = s^2$.

Property 3. The number A of square units of the interior of a parallelogram is the product of the numbers of linear units b and h in the base and height. Thus $A = bh$.

Property 4. The number A of square units of area in the interior of a triangle is $\frac{1}{2}$ the product of the numbers of linear units of measure in the base and height. $A = \frac{1}{2}bh$.

Property 5. The number A of square units of area in the interior of an equilateral triangle is: $A = \frac{\sqrt{3}}{4} \cdot s^2$.

Property 6. The number A of square units of area in the interior of a regular polygon is $\frac{1}{2}$ the product of the numbers of linear units in its perimeter and the distance from the center to each side.

Property 7. If r is the number of linear units in the radius of a circle and A the number of square units of area in its interior then $A = (\frac{1}{2}r)(a\pi r)$ or

$$A = \pi r^2 .$$

Property 8. If a line is perpendicular to two distinct intersecting lines in a plane, it is perpendicular to the plane.

Property 9. The shortest segment having one end at a given point Q outside a plane and the other on a given plane r not containing the point Q is a segment perpendicular to the plane r.

Property 10. If two planes are parallel the distances from different points of one plane to the other plane are all the same.

Property 11. The numbers of square units of surface area in the interiors of the bases and lateral faces of a right rectangular prism is $S = 2 [lw + h + wh]$.

Property 12. The number (V) of cubic units of volume in the interior of a right rectangular prism is the product of the number (B) of square units of area in the interior of the base and the number (h) of linear units in the height. Thus $V = Bh$.

Property 13. The number of cubic units of volume V in the interior of a cube whose edges are s linear units in length is given by $V = s^2 \cdot s = s^3$.

Property 14. The number of square units S of surface area in the faces of a cube is $6s^2$.

Property 15. In general the volume (V) of the interior of a right prism with polygonal bases is the product of the number of square units (B) in the interior of the base and the number of linear units (h) in the height. Thus $V = Bh$.

Property 16. The number V of cubic units of volume in the interior of a right circular cylinder is the product of the number of square units (B) in the interior of the circular base and the number of linear units (h) in the height. Thus $V = \pi r^2 h$.

Property 17. If r and h are the numbers of linear units in the radius of the base and the height of a right circular cone and S the number of square units in the total area, then $S = 2\pi rh + 2\pi r^2$ or $2\pi r(r+h)$.

Property 18. If two prisms have congruent bases and equal heights they have equal volumes.

Property 19. The number of cubic units of volume in the interior of any prism is obtained from the formula, $V = Bh$.

Property 20. If two pyramids have congruent bases and equal heights they have equal volumes.

Property 21. The volume of a pyramid is one third that of a prism whose base is congruent to the base of the prism and whose height is the same as that of the prism.

Property 22. The number V of cubic units of volume in a pyramid is given by the formula, $V = 1/3\ Bh$, where B stands for the number of square units of area in the base and h the number of linear units in the height.

Property 23. The volume of the interior of a cone is one third that of a cylinder of the same height and whose base has the same radius.

Property 24. If r is the number of linear units in the radius of the base of a cone and h the number of linear units in the height, the number V of cubic units in the volume of its interior is given by the formula, $V = 1/3\pi r^2h$. Since πr^2 is actually the number of square units B of area in the base, the formula could be written as $V = 1/3Bh$.

Property 25. If the slant height of a right circular cone is s units and the radius of its base r units, the number L of square units in its lateral area is given by the formula,

$$L = \pi r s \quad .$$

UNIT 10

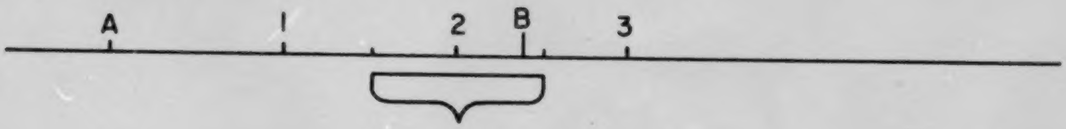
RELATIVE ERROR

10-1. Greatest Possible Error

When you use numbers to count separate objects you need only counting numbers. In counting we set up a one-to-one correspondence between the objects counted and the members of the set of counting numbers. When you count the number of people in a classroom you know the result will be a counting number; there may be exactly 11, but there cannot be $11\frac{1}{4}$ or $10\frac{1}{2}$. If there are a great many people, or if you are not sure you have counted correctly, you may say there are "about 300", rounding the number to the nearest hundred.

When you measure something, the situation is different. When you have measured the length of a line segment with a ruler divided into quarter-inches, the end of the segment probably fell between two quarter-inch marks, and you had to judge which mark appeared closer. Even if the end seemed to fall almost exactly on a quarter-inch mark, if you had looked at it through a magnifying glass you would probably have found that there was a difference. And if you had then changed to a ruler with the inches divided into sixteenths, you might have decided that the end of the segment was nearer to one of the new sixteenth-inch marks than to a quarter-inch mark.

In any discussion of measurement we assume proper use of instruments. Improper use of instruments can occur through ignorance, or carelessness. These mistakes can be corrected by learning how the instrument works and by careful inspection during the measurement process. Scientists and mathematicians agree that measurement cannot be considered exact, but only approximate. The important thing to know is just how inexact a measurement is, and to state clearly how exact it is.



Look at the line above, which shows a scale divided into one-inch units. The zero point is labeled "A", and point B is between the 2-inch mark and the 3-inch mark. Since B is clearly closer to the two-inch mark, we may say that the measurement of segment $\overline{AB}$ is 2 inches. However, any point which is more than $1\frac{1}{2}$ inches from A and less than $2\frac{1}{2}$ inches from A would be the endpoint of a segment whose length, to the nearest inch, is also 2 inches. The mark below the line shows the space within which the endpoint of a line segment 2 inches long (to the nearest inch) might fall. The length of such a segment might actually be almost $\frac{1}{2}$ inch less than 2 inches, or almost $\frac{1}{2}$ inch more than 2 inches. We therefore say that, when a line segment is measured to the nearest whole inch the "greatest possible error" is $\frac{1}{2}$ inch. This does not mean that you have made a mistake (or that you have not). It simply means that, if you measure properly to the nearest whole inch, any measurement more than $1\frac{1}{2}$ inches and less than $2\frac{1}{2}$ inches will be correctly reported in the same way, as 2 inches. Consequently such measurements are sometimes stated as $2 \pm \frac{1}{2}$ inches. (The symbol "±" is read "plus or minus".) The greatest possible error of this measure is $\frac{1}{2}$ inch. When we state a measurement as 2 inches in this unit we mean $2 \pm \frac{1}{2}$ inches, 2 ± 0.5 inches. In the everyday world this is often not the case, therefore in industrial and scientific work the greatest possible error should be specifically stated, for example a measurement should be given as 2 ± 0.05 inches or 2 ± 0.005 inches, never simply as 2 inches.

Often in business and industry the term tolerance is used. Tolerance means the greatest error which is allowed. The tolerance might be set by the person who is purchasing a certain manufactured product or by the operation of a machine. For instance, an automobile manufacturer might specify that the cylinders of an engine should have a diameter of 5 inches with a tolerance of one-thousandth of an inch. This means the diameter cannot vary more than 0.001 inch from 5 inches; the dimension would be given as 5 ± 0.001 inches. On the other hand, a producer of water pumps might demand a tolerance different from 0.001 inch. Laws often specify tolerance for instruments in commercial use like scales for weighing. Scales are allowed to vary within certain limits. Court cases are sometimes decided on the basis of tolerances allowed in the calibration of police car speedometers.

Exercises 10-1

1. Draw a line and mark on it a scale with divisions of $\frac{1}{4}$ inch. Mark the zero point C. Place a point between $1\frac{2}{4}$ and $1\frac{3}{4}$, but closer to $1\frac{3}{4}$, and call the point D. How long is $\overline{CD}$ to the nearest $\frac{1}{4}$ inch?
2. Between what two points on the scale must D lie if the measurement, to the nearest $\frac{1}{4}$ inch, is to be $1\frac{3}{4}$ inch? How far from $1\frac{3}{4}$ is each of these points?

3. Why may this measurement of $\overline{CD}$ be stated "$1\frac{3}{4} \pm \frac{1}{8}$" ?

4. (a) The measurement of a line segment was stated to be $1\frac{1}{2}$ inches. This segment must have been measured to the nearest ____________ of an inch.

   (b) The endpoint of the segment must have fallen between 1______ and 1______.

   (c) The measurement might be stated as $1\frac{1}{2} \pm$ _______ inches.

   (d) The "greatest possible error" in the measurement of this segment is _________.

5. The measure of a line segment was stated as $2\frac{5}{16} \pm \frac{1}{32}$.

   (a) Between what marks on the scale must the end of this segment lie?

   (b) What is the greatest possible error?

6. If a measurement is stated to be $\frac{5}{16}$ inch, this means that the measurement was made to the nearest ______ of an inch, and the unit is _________ inch.

7. When you measure to the nearest $\frac{1}{4}$ inch, what is the "greatest possible error"?

8. A meter stick is divided into centimeters and tenths of a centimeter. A line segment was measured with such a scale, and stated to be $3\frac{7}{10}$ centimeters.

   (a) What was the unit of measurement?

   (b) State the measure $3\frac{7}{10} \pm$ ________.

   (c) What was the greatest possible error?

9. Scientists sometimes measure to the nearest $\frac{1}{100}$ of a centimeter.

10. The "greatest possible error" in a measurement is always what fractional part of the unit used?

10-2. Precision and Significant Digits

Consider the two measurements, $10\frac{1}{8}$ inches and $12\frac{1}{2}$ inches. As commonly used, these measurements do not indicate what unit of measurement was used. Suppose that the unit for the first measurement is $\frac{1}{8}$ inch, and the unit for the second measurement is $\frac{1}{2}$ inch. Then we say that the first measurement is more precise than the second, or has greater precision. Notice also that the greatest possible error of the first measurement is $\frac{1}{2}$ of $\frac{1}{8}$ inch, or $\frac{1}{4}$ inch. The greatest possible error is less for the first measurement, than for the second measurement. Hence the more precise of two measurements is the one made with the smaller unit, and for which the greatest possible error is therefore the smaller.

It is very important that measurements be stated so as to show correctly how precise they are. In this unit we adopt the convention that the denominator of the fractional part of a measurement indicates the unit of measurement which was used. If a line segment is measured to the nearest $\frac{1}{8}$ inch, and the measurement is $2\frac{6}{8}$ inches, we shall not change the fraction to $\frac{3}{4}$, for that would make it appear that the unit was $\frac{1}{4}$ inch, rather than $\frac{1}{8}$ inch. If a line segment is measured to the nearest $\frac{1}{4}$ inch, and the measurement is closer to 3 inches than to $2\frac{3}{4}$ or $3\frac{1}{4}$ inches, we shall state it to be $3\frac{0}{4}$, so that it is clear that the unit used is $\frac{1}{4}$ inch.

Usually scientific measurements are expressed in decimal form. For instance, it is known that one meter (a unit in the metric system of measures) is about 39.37 inches. This means that a meter is closer to 39.37 inches than it is to 39.38 inches or 39.36 inches. In other words, one meter lies between 39.375

inches and 39.365 inches.

The measure of 39.37 includes 4 significant digits. They are significant in that they tell us the precision of our measurement. The place value of the last significant digit to the right indicates the precision, in this case one hundredth of an inch.

All non-zero digits are significant. A zero may or may not be significant. Zeros are significant when they are between non-zero digits as in numerals like 2007 (4 significant digits), 80,062 (5 significant digits), and 3.08 (3 significant digits). Zeros are not significant in numerals such as 0.008 and 0.026 because the zeros are used only to fix the decimal point.

If we were told that something is 73,000 feet long, it is not clear whether or not the zeros at the end are significant and actually indicate the precision. There is doubt about the precision of such a measurement. The unit of measurement may have been 1,000 feet, 100 feet, 10 feet, or 1 foot. In a case like this, a zero is sometimes underlined to show how precise the measurement is. For example, $73,\underline{0}00$ (3 significant digits) means that the measurement is precise to the nearest 100 feet, $73,0\underline{0}0$ (4 significant digits) means that the measurement is precise to the nearest 10 feet, and $73,00\underline{0}$ (5 significant digits) means that the measurement is precise to the nearest foot. If no zero is underlined, we understand that the measurement was made to the nearest 1000 feet. If a measurement is stated as 5.640 feet we understand, without underlining the zero, that it is significant and that the unit is one thousandth of a foot, for otherwise the zero would not be written at all.

When a number is written in scientific notation, all of the digits in the first factor are significant. For example, the measurement 2.99776×10^{10} cm/sec, for the velocity of light, has 6 significant digits; the measurement 2.57×10^{-9} cm for the radius of the hydrogen atom, has 3 significant digits; the measurement for the national debt in 1957: 2.8×10^{11} dollars, has 2 significant digits. 4.800×10^{8} has 4 significant digits. In the last case, the two final zeros are significant. Were they not, the number should have been written as 4.8×10^{8}.

Exercises 10-2

1. Suppose you measured a line to the nearest hundredth of an inch. Which of these numbers states the measurement best?

   3.2 inches   3.20 inches   3.200 inches

2. Suppose you measured to the nearest tenth of an inch. Which of these numbers should you use to state the result?

   4 inches   4.0 inches   4.00 inches

3. Tell which measurement in each pair has the greater precision.

   (a) 5.2 feet,   2 1/4 feet
   (b) 0.68 feet,   23.5 feet
   (c) 0.235 inches,   0.146 inches

4. What is your age to the nearest year, that is, what is your nearest birthday -- tenth, eleventh, twelfth......?
   All of you who say "13" must be between _____ and _______ years old.

5. (a) For each measurement below tell the place value of the last significant digit.

   (b) Tell the greatest possible error of the measurements.

(a) 52700 feet (b) $527\underline{0}$ feet (c) $527\underline{0}0$ feet

(d) 52.7 feet (e) 0.5270 feet (f) 527.0 feet

6. (a) Which of the measurements in Problem 5 is the most precise?

   (b) Which is the least precise?

   (c) Do any two measurements have the same precision?

7. Show by underlining a zero the precision of the following measurements:

   (a) 4200 feet measured to the nearest foot.

   (b) 23,000 miles, measured to the nearest hundred miles.

   (c) 48,000,000 people, reported to the nearest ten-thousand.

8. Tell the number of significant digits in each measurement:

   (a) 520 feet (b) 32.46 in. (c) 0.002 in. (d) 403.6 ft.

   (e) $25{,}8\underline{0}0$ ft. (f) 0.0015 in. (g) 38.90 ft. (h) 0.0603 in.

9. How many significant digits are in each of the following:

   (a) 4.700×10^5 (d) 6.70×10^{-4}

   (b) 4.700×10^4 (e) 4.7000×10

   (c) 4.7×10^{15} (f) 2.8×10^9

10-3. Relative Error, Accuracy and Percent of Error

While two measurements may be made with the same precision (that is, with the same unit) and therefore with the same greatest possible error, this error is more important in some cases than in others. An error of 1/2 inch in measuring your height would not be very misleading, but an error of 1/2 inch in measuring your nose would be misleading. We can get a measure of the importance of the greatest possible error by comparing it with the measurement. Consider these measurements and their greatest possible errors:

4 in. ± 0.5 in.: 58 in. ± 0.5 in.

Since these measurements are both made to the nearest inch, the greatest possible error in each case is 0.5 inch. If we divide the greatest possible error by each of the measurements we get these results:

$$\frac{0.5}{4} = \frac{5}{40} = 0.125$$

$$\frac{0.5}{58} = \frac{5}{580} = 0.0086$$

The quotients 0.125 and 0.0086 are called relative errors. The relative error of a measurement is defined as the quotient of the greatest possible error by the measured value.

$$\text{Relative error} = \frac{\text{greatest possible error}}{\text{measured value}}$$

Percent of error is the relative error expressed as a percent. In the above two examples the percent of error is 12.5% and 0.86%.

The measurement with a relative error of 0.0086 (0.86%) is more accurate than the measurement with a relative error of 0.125 (12.5%). By definition a measurement with a smaller relative error is said to be more accurate than one with a larger relative error.

The terms accuracy and precision are used in industrial and scientific work in a special technical sense even though they are often used loosely and as synonyms in everyday conversation. Precision is the length of the unit of measurement, which is twice the greatest possible error while accuracy is the relative error. For example, 12.5 pounds and 360.7 pounds are equally precise, that is, precise to the nearest 0.1 of a pound (greatest possible error in each case is 0.05 pound). The two measurements do not possess the same accuracy. The second measurement is more accurate. You should verify the last statement by computing the relative errors in each case and comparing them.

An astronomer, for example, making a measurement of the distance to a galaxy may have an error of a trillion miles (1,000,000,000,000 miles) yet be far more accurate than a machinist measuring the diameter of a steel pin to the nearest 0.001 inch.

Again, a measure indicated as 3.5 inch and another as 3.5 feet are equally accurate but the first measure is more precise. Show this!

Exercises 10-3

In all computation express your answer so that it includes two significant digits.

1. State the greatest possible error for each of these measurements.
   (a) 52 ft. (b) 4.1 in. (c) 2580 mi. (d) $36\underline{0}$ ft.
   (e) 7.03 in. (f) 0.006 ft. (g) 54,000 mi. (h) $54{,}0\underline{0}0$ mi.
2. Find the relative error of each measurement in Problem 1.
3. Find the greatest possible error and the percent of error for each of the following measurements.
   (a) 9.3 ft. (b) 0.093 ft. (c) 930 ft. (d) 93,000 ft.
4. What do you observe about your answers for Problem 3? Can you explain why the percents of error should be the same for all of these measurements?
5. Find the precision of the following measurements.
   (a) 26.3 ft. (b) 0.263 ft. (c) 2630 ft.
   (d) 51,000 mi. (e) 5.1 ft. (f) 0.051 in.

6. How many significant digits are there in each of these measurements?

(a) 52.1 in. (b) 52.10 in. (c) 3.68 in. (d) 368.0 in.

7. Find the relative error of each of the measurements in Problem 6.

8. From your answers for Problems 6 and 7, can you see any relation between the number of significant digits in a measurement and its relative error? What is the relation between the number of significant digits in a measurement and its accuracy?

9. Without computing, can you tell which of the measurements below has the greatest accuracy? Which is the least accurate?

23.6 in. 0.043 in. 7812 in. 0.2 in.

10. Arrange the following measurements in the order of their precision (from least to greatest).

(a) $36\frac{1}{2}$ in., $27\frac{0}{16}$ in., $32\frac{3}{8}$ in., $46\frac{2}{7}$ in., $22\frac{1}{4}$ in.

(b) 4.62 in., 3.041 in., 3 in., 82.4 in., 0.3762 in.

*11. Arrange the following measurements in order of their accuracy (from least to greatest):

6 ft. ($\pm\frac{1}{2}$ ft.), 3.2 in. (± 0.005 in.), 7.2 miles (± 0.05 mile), $3\frac{1}{2}$ in. ($\pm\frac{1}{8}$ in.), 3 yd. 4 in. ($\pm\frac{1}{4}$ in.).

12. Count the number of significant digits in each of the following.

(a) 43.26
(b) 4,607
(c) 32.004
(d) 0.0062
(e) 0.6070
(f) 0.0030
(g) 4,0030
(h) 0.03624
(i) 76,000
(j) 43,000
(k) 0.036
(l) 200.00004

13. Express the following in scientific notation.

(a) 463,000,000
(f) 0.0000400

(b) 327,000

(c) 0.000462

(d) 32.004

(e) 2

(g) 36.8×10^{5}

(h) 0.8×10^{-7}

(i) 72 billion

14. By inspection arrange the following, by letter, in order of their magnitude, from least to greatest.

(a) 3.6×10^{5}

(b) 3.5×10^{8}

(c) $4. \times 10^{-6}$

(d) 3.527×10^{8}

(e) 3.5×10^{-12}

(f) 4.1×10^{6}

(g) 3.527×10^{2}

(h) 3.55×10^{8}

(i) 3.4×10^{-7}

(j) 3.39×10^{-8}

15. BRAINBUSTER. A master machinist measures a $3\frac{1}{2}$ inch piston head to the nearest 0.0001 inch while an astronomer measures by the parallax, the distance to Canis Major (the star Sirius) as correct to the nearest 10,000,000 miles. The distance to Sirius is 8.6 light years (1 light year $\approx 6 \times 10^{12}$ miles). Which measurement is more accurate?

10-4. Adding and Subtracting Measurements

Since measurements are never exact, the answers to any questions which depend on those measurements are also approximate. For instance, suppose you measured the length of a room by making two marks on a wall, which you called A and B, and then measuring the distances from the corner to A, from A to B, and from B to the other corner. Measurements such as these which are to be added, should all be made with the same precision. Suppose, to the nearest fourth of an inch, the measurements were $72\frac{1}{4}$ inches, $40\frac{2}{4}$ inches, $22\frac{3}{4}$ inches. You would add these amounts to get $135\frac{2}{4}$ inches.

Of course, the distances might have been shorter in each case. They could have been almost as small as $72\frac{1}{8}$, $40\frac{3}{8}$, and $22\frac{5}{8}$ in which case the distance would have been almost as small as $135\frac{1}{8}$ inches, which is three-eighths of an inch less than $135\frac{2}{4}$. Also, each distance might have been longer by nearly one-eighth of an inch, in which case the total length might have been almost three-eighths of an inch longer than $135\frac{2}{4}$. The greatest possible error of a sum is the sum of the greatest possible errors. If we were adding measures of 37.6, 3.5, and 178.6, the greatest possible error of the sum would be 0.15. The result of this addition could be shown as 219.7 ± 0.15.

Computation involving measurements is very important in today's world. Many rules have been laid down giving the accuracy or precision of the results obtained from computation with approximate measurements. Too many rules, however, might create confusion and would never replace basic knowledge of approximate data. If the meaning of greatest possible error and of relative error is understood, the accuracy or precision of the result of computation with approximate data can usually be found by applying common sense and judgment. Common sense would tell us that with a large number of measurements the errors will, to a certain extent, cancel each other.

The general principle is that the sum or difference of measurements cannot be more precise than the least precise element involved. Therefore to add or subtract numbers arising from approximations, first round each number to the unit of the least precise number and then perform the operation.

As we have seen, the greatest possible error of a sum (or difference) of several measurements is the sum of the greatest possible errors of the measurements involved. (To estimate the expected error of a sum, taking into account the way the errors would often cancel each other, we need to use some ideas of probability.)

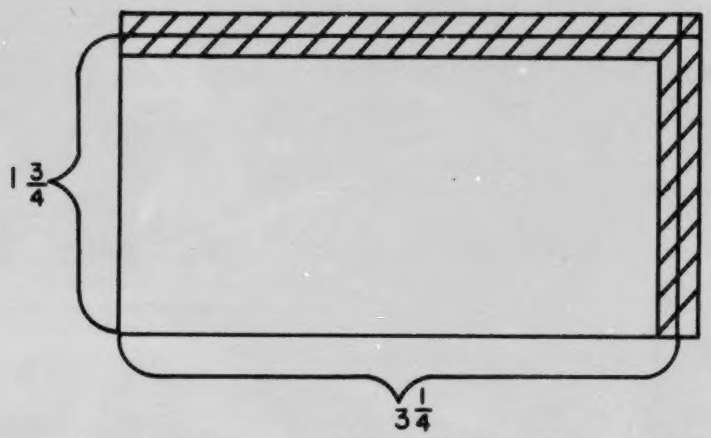
Exercises 10-4

1. Find the greatest possible error of the sums of these measures:
   - (a) $5\frac{1}{2}$ in., $6\frac{1}{2}$ in., $3\frac{0}{2}$ in.
   - (b) $3\frac{1}{4}$ in., $6\frac{1}{2}$ in., 3 in.
   - (c) 4.2 in., 5.03 in.
   - (d) 42.5 in., 36.0 in., 49.8 in.
   - (e) 0.004 in., 2.1 in., 6.135 in.
   - (f) $2\frac{3}{4}$ in., $1\frac{5}{16}$ in., $3\frac{3}{8}$ in.
2. Add the following measures:
   - (a) 42.36, 578.1, 73.4, 37.285, 0.62
   - (b) 85.42, 7.301, 16.015, 36.4
   - (c) 9.36, 0.345, 1713.06, 35.27
3. Subtract the following measures:
   - (a) 7.3 - 6.28
   - (b) 735 - 0.73
   - (c) 5430 - 647

10-5. Multiplying and Dividing Measurements

You know that the area of a rectangle is found by multiplying the number of units in the length by the number of the same units in the width. Suppose that the dimensions of a rectangle are $3\frac{1}{4}$ inches and $1\frac{3}{4}$ inches. Since the measuring was done to the nearest $\frac{1}{4}$ inch, the measurements can be stated as $3\frac{1}{4} \pm \frac{1}{8}$ and

$1\frac{3}{4} \pm \frac{1}{8}$. This means that the length might be almost as small as $3\frac{1}{8}$ inches and the width almost as small as $1\frac{5}{8}$ inches. The length might be almost $3\frac{3}{8}$ inches and the width almost $1\frac{7}{8}$ inches.



Look at the drawing to see what this means. The outside lines show how the rectangle would look if the dimensions were as large as possible. The inner lines show how it would look if the length and width were as small as possible. The shaded part shows the difference between the largest possible area and the smallest possible area with the given measurements.

To find these areas, we multiply $3\frac{1}{8} \times 1\frac{5}{8}$ to find the smallest possible area, and $3\frac{3}{8} \times 1\frac{7}{8}$ to find the largest possible area.

$$3\frac{1}{8} \times 1\frac{5}{8} = \frac{25}{8} \times \frac{13}{8} = \frac{325}{64} = 5\frac{5}{64}$$

$$3\frac{3}{8} \times 1\frac{7}{8} = \frac{27}{8} \times \frac{15}{8} = \frac{405}{64} = 6\frac{21}{64}.$$

These results show that there is a difference of more than 1 square inch in the two possible areas.

If we find the area by using the stated length and width, we find: $3\frac{1}{4} \times 1\frac{3}{4} = \frac{13}{4} \times \frac{7}{4} = \frac{91}{16} = 5\frac{11}{16}$.

However, since we have seen that the area might be either larger or smaller than this number of square inches, it would not be correct to give the result in this way, which means (by our agreement) that the area has been found to the nearest 16th of a square inch. The area could be as much as $\frac{41}{64}$ square inches greater or $\frac{39}{64}$ square inches less than $5\frac{11}{16}$. Thus, it could be given as $5\frac{11}{16} \pm \frac{41}{64}$ square inches.

The statement of a rule for multiplication of approximate data in fractional form would be difficult. However, when data are expressed in decimal form, a rough guide can be employed for finding a satisfactory product. The product of two numbers arising from approximation has as many significant digits as there are in the factor with the fewer significant digits. For example: The area of the rectangle 10.4 cm. by 4.7 cm. would be stated as 49 sq. cm.

$$\begin{array}{r} 10.4 \\ 4.7 \\ \hline 728 \\ 416 \\ \hline 48.88 \end{array}$$

The area is 49 sq. cm.

If one of the two factors contains more significant digits than the other, round off the factor which has more significant digits so that it contains only one more significant digit than the other factor.

If we wish to find the circumference of a circle with the diameter equal to 5.1 we use $\pi \approx 3.14$ as in the above rule.

Division is defined by means of multiplication. Therefore it is reasonable to follow the procedures used for multiplication in doing divisions involving approximate data.

When a multiplication or division involves an exact number as 2 in the formula for the circumference of a circle ($C = 2\pi r$), the approximate number always has the smaller number of significant digits. We ignore the exact number in determining the significant digits in the answer.

Exercises 10-5

1. Suppose a rectangle is $2\frac{1}{2}$ inches long and $1\frac{1}{2}$ inches wide. Make a drawing of the rectangle. Show on the drawing that the length is $2\frac{1}{2} \pm \frac{1}{4}$ and the width $1\frac{1}{2} \pm \frac{1}{4}$. Then find the largest area possible and the smallest area possible, and find the difference, or uncertain part. Then find the area with the measured dimension, and find the result to the nearest $\frac{1}{2}$ square inch.
2. Multiply the following approximate numbers:
   (a) 4.1×36.9
   (b) 3.6×4673
   (c) $3.76 \times (2.9 \times 10^4)$
3. Divide the following approximate numbers:
   (a) $3.632 \div .83$
   (b) $0.000344 \div 0.000301$
   (c) $(3.14 \times 10^6) \div 8.006$
4. Find the area of a rectangular field which is 835.5 rods long and 305 rods wide.
5. The circumference of a circle is stated $C = \pi d$, in which d is the diameter of the circle. If π is given as 3.141593, find the circumferences of the following circles whose diameters

are:

(a) 3.5 in.

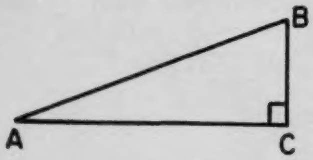
(b) 46.36 ft.

(c) 6 miles.

6. A machine stamps out parts each weighing .625 lb. How much weight is there to 75 of these parts?

7. Assuming that water weighs 62.5 lb. per cu. ft., what is the volume of 15,610 lbs?

8. (a)



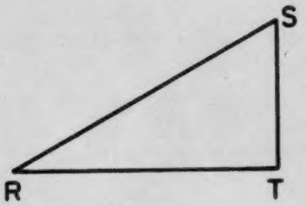
Given: $m(\angle A) = 22^\circ$

$\sin 22^\circ = 0.3746$

$AB = 34 \pm 0.5$ ft.

Find: BC

(b)



Given: $m(\angle S) = 73^\circ$

$\tan 73^\circ = 3.2709$

$ST = 67 \pm 0.5$ ft.

Find: RT

There are many rough rules for computing with approximate data but they have to be used with a great deal of common sense. They don't work in all cases. The modern high speed computing machine which adds or multiplies thousands of numbers per second has to have special rules applied to the data which are fed to them. Errors involved in rounding numbers add up or disappear in a very unpredictable fashion in these devices. As a matter of fact "error theory" as applied to computers is an active field of research today for mathematicians.

UNIT 11

PERMUTATIONS AND COMBINATIONS

11-1. The Pascal Triangle

Five girls form a club. Their names are Alice, Betty, Carol, Doris, and Ellen; sometimes we shall call them by their initials A, B, C, D, E. The first order of business in the club is to choose a refreshment committee. It is agreed that the committee should have three members. How many possibilities for the membership on the committee do you believe there are?

One possibility would be a committee consisting of Betty, Carol, and Ellen. We might abbreviate this possible committee by the symbol { B,C,E }.

Class Discussion Exercises 11-1a

1. Another possible membership list is Alice, Doris, Ellen. Write the abbreviated symbol for this case.
2. Give the regular names of the girls in the committee { B,E,A }.
3. Does the committee {B,E,A } have the same members as the committee {E,A,B }?
4. Give two other symbols, each of which names the committee mentioned in Question 3.
5. Make a list of all the possible committees.
6. How many committees are in your list?
7. Of how many of these committees is Doris a member?
8. Compare the number of committees of which Betty is a member and the number of those which include Doris.
9. (Answer this question without doing any more counting.) How

many of the committees do not include Alice?

10. What is the ratio of the number of committees including Ellen to the number of possible committees? (Did you answer this question without further counting?)

11. How many committees have both Alice and Carol as members? We may easily answer this last question without looking at our list of all the possible committees. We observe that since a committee {A,C,?} has two members specified, then there is only one vacancy to be filled. How many possible choices are there for third member? Thus three of the ten possible committees include both Carol and Alice.

12. What is the ratio of the number of possible committees including both Betty and Ellen to the number of committees including Betty?

Whenever three girls are chosen for a special purpose, such as membership on a committee, then the remaining two girls have also been chosen -- chosen, in the sense that they will not serve on this particular committee. In other words, the selection of a committee, in effect, separates the club members into two groups. One method for selecting the membership of a committee is to decide which girls will not serve. For example, if it is decided that a committee should not include Carol and Doris, then we know that the committee is {A,B,E}.

13. Name the committee determined by the condition that Alice and Ellen have been chosen to be not members.

14. Which two girls are picked as not being in {E,B,C}?

The selection of a committee of three girls also means a choice of another committee with two members, namely the other two of the five club girls. For example, the selection of {E,B,C} determines the two-member committee {A,D}.

15. Since there are ten possible committees with three members each, how many possible committees with two members each are there?
16. Since six of the possible three-girl committees include Betty, how many of the possible two-girl committees exclude Betty?
17. How many of the possible two-girl committees include Carol?
18. Find the answer to Question 17, using the method of filling the vacancy in {C,?}.

Exercises 11-1a

1. From the club {A,B,C,D,E}, one possible committee with four members is {A,B,D,E}.
   (a) Make a list of all the possible committees, each with four members.
   (b) How many girls in the club are excluded from a committee with four members?
   (c) What relationship is there between the number of possible committees with four members each and the number of possible committees with one girl each?
   (d) Make a list of all the possible committees with one girl each.
   (e) How many committees are there with all five girls as members?
2. A club has four members whom we may call K,L,M,N.
   (a) How many committees in this club have four members?

(b) How many possible committees with one member each are there?

(c) Name each of the one-member committees.

(d) To each one-member committee there corresponds, in a natural way, a committee with three members. What is that natural way?

(e) Use parts (c) and (d) to make a list of the possible committees with three members each.

(f) Make a list of the possible committees with two members each.

3. (a) Make a list of all the possible committees, and note how many committees there are of each size, in a club with three members.

   (b) Do the same for a club with two members.

   (c) Do the same for a club with just one member.

4. A family would enjoy each of four vacation spots. It is decided to choose two of the four and spend part of the vacation time at each of the two. How many possible choices are there for the pair of vacation places?

5. The refrigerator holds two cartons of ice cream. The dairy has five flavors, and the family always likes to buy two different flavors. How many times can the family go to the dairy and bring home a different pair of flavors?

Let us make a table showing the number of possible committees with a given number of members from a club with a given number of members. This table will summarize several of the results we have obtained in problems. In a club with five members, there are 5 possible committees with one member, 10 committees with two members, 10 with three, 5 with four, 1 with five. The selection of a committee which includes all five club members (sometimes referred to as the "committee of the whole") means that there are zero club members not serving. Thus we may balance our table by saying that there is 1 possible committee with zero members. (You may wish to compare this agreement with the remark that there is just one empty set.)

If we arrange our data according to increasing size of committees, we have

1 5 10 10 5 1

These six numbers tell us how many possible committees of various sizes can be chosen from a club membership of five.

The same type of data, for a club membership of four, is the following.

1 4 6 4 1

Be sure that you understand the significance of each of these five entries.

Class Discussion Exercises 11-1b

1. In particular, what does the last 1 in the data 1,4,6,4,1 mean?
2. What does the first 1 mean?
3. What is the corresponding data for a club membership of three?
4. How can you interpret the data

   1 2 1?
5. What data of this type do we have for a club with only one member?

Let us collect together into a table the data for the various sizes of club. Each row of the table shows the information for a certain size of club.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Let us examine again the entry in the table telling how many possible committees of three girls each can be named from the club { A,B,C,D,E } whose members' regular names are Alice, Betty, Carol, Doris, Ellen. In the table the entry is which of the 10's?

A committee of three may include Ellen or it may not. We will study these two cases in more detail.

6. How many possible committees with three members include Ellen?
7. A committee including Ellen is of the type { E,?,? }. How many vacancies appear? From how many girls can these vacancies be filled?
8. In view of Question 7, compare the answer to Question 6 with the number of committees with two members from a club of four members.
9. How many possible committees with three members exclude Ellen?
10. A committee excluding Ellen is of the type { ?,?,? } where no blank may be filled with E. How many vacancies appear? From how many girls can these vacancies be filled?
11. In view of Question 10, the answer to Question 9 is the same as the number of committees with ___________ (how many?) members

from a club of ____________ (how many?) members.

By encircling we show in the table below the three entries we have been studying.

$$\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The entry 10 is the sum of the two numbers, 6 and 4, nearest it on the preceding line.

Exercises 11-1b

1. Check that (except for the entries 1) every entry in the table is the sum of the two numbers nearest it on the preceding line.
2. Explain why the entry 6 in the fourth line *ought to be* the sum of the two numbers, 3 and 3, above it.
3. Explain why the third entry in the third line (the right-hand 3) *ought to be* the sum of the numbers, 2 and 1, above it.

The table we have been studying is a part of the array known as the Pascal triangle. (The French mathematician Pascal, seventeenth century, contributed to geometry and the theory of probability.) The table would resemble even more an equilateral triangle if we supplied a vertex at the top; this is sometimes done, but we shall not be concerned with it. In our version of the Pascal triangle, the first, second, third, fourth, fifth rows show the numbers of possible committees from a club of one, two, three, four, five members respectively. What do you expect for a club with six members? The Pascal triangle has a sixth row, which we have not yet found.

Class Discussion Exercises 11-1c

1. What should be the left-hand entry in the sixth row?
2. From a club with six members, how many possible committees with one member each are there?
3. Is the second entry (from the left) in the sixth row the sum of the two numbers nearest it in the fifth row?
4. Complete the sixth row.

Exercises 11-1c

1. A club has six members, Alice, Betty, Carol, Doris, Ellen, Freda.
   (a) Some of the possible committees with two members are {A,B}, {A,C}, {C,E}. Make a list of all fifteen of these committees. (Write your list down the page, using fifteen rows.)
   (b) Make a list of all the possible committees with four members. Do this on the right-hand side of your answer to part (a). Specifically, for each committee in the list for part (a), beside it write the committee whose four members are excluded from the committee with two girls. As an example, one line on your answer sheet will be:

   {A,C} {B,D,E,F}

   (c) After you have written your list of committees with four members each, does the number of these possible committees agree with the number obtained from the fifth row of the

Pascal triangle? (How is this number obtained from the fifth row?)

(d) Make a complete list of all possible committees with three members each.

(e) Does the number of committees listed in part (d) agree with the number obtained from the fifth row of the Pascal triangle?

2. Find the seventh row of the Pascal triangle.
3. Find the eighth row of the Pascal triangle.
4. What are the first two entries (on the left) in the twenty-third row of the Pascal triangle?
5. What are the last two entries (on the right) in the fifty-seventh row of the Pascal triangle?
6. In the first row of the Pascal triangle the sum of the numbers is $1 + 1 = 2$. The sum of the numbers in the second row is $1 + 2 + 1 = 4$.

   (a) Find the sum of the numbers in each of the third, fourth, and fifth rows.

   (b) For each of these sums, find its complete factorization (as a product of primes).

   (c) Do you see any pattern in these successive sums? How is each of these sums related to the preceding one?

   (d) Can you predict the sum of the numbers in the sixth row?

   (e) Check your prediction by adding the numbers in the sixth row and comparing the sum with the predicted sum. (In case they do not agree, what should you do?)

   (f) Use the pattern to predict the sum of the numbers in the seventh row, and then check by adding the numbers.

(g) What do you believe is the sum of the numbers in the twelfth row of the Pascal triangle?

(h) How many entries are there in the two hundredth row of the Pascal triangle? What is the sum of all these numbers?

Let us now study the third entry from the left in each row of the Pascal triangle. Of course the first row does not have a third entry. From then on, we have the numbers 1,3,6,10,15,21, ··· . Do you see any pattern in this list of numbers?

As you read this paragraph, keep track of the mentioned numbers from the Pascal triangle. The entry 3 in the list 1,3,6,10,15,··· may be considered the sum of 2 and 1 from the second row of the table. The next entry 6, in the list is the sum of two numbers from the third row, namely 3 (telling which row) and 3 (the preceding entry in the list under study). Since $6 = 3 + 3$ and since $3 = 2 + 1$, we observe that $6 = 3 + 2 + 1$. The fourth entry 10, in our list is, from the table, the sum of 4 and 6; and $6 = 3 + 2 + 1$; thus $10 = 4 + 3 + 2 + 1$. Check that the next entry in our list is $5 + 4 + 3 + 2 + 1$. Explain why 15, the third entry in the sixth row of the Pascal triangle, should be the same as $5 + 4 + 3 + 2 + 1$. Explain the number 21 as the third entry in the seventh row.

Class Discussion Exercises 11-1d

1. In which row is the number $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$ the third entry?
2. Give a name for the third entry in the sixteenth row.
3. Give a name for the next to the next to the last entry in

the ninetieth row.

4. A sum such as $15 + 14 + 13 + \cdots + 3 + 2 + 1$ is the same as $1 + 2 + 3 + \cdots + 13 + 14 + 15$. Twice this sum is $16 + 16 + 16 + \cdots + 16 + 16 + 16$. Why?

5. How many 16's are indicated in the last Question? What is their sum?

6. What is a simple name for the number $15 + 14 + \cdots + 2 + 1$?

7. What is the best name for the number in Question 2?

Exercises 11-1d

1. How many possible committees with two members each can be chosen from a club with 30 members?

2. Let n be a counting number greater than 2. Then in the n^{th} row of the Pascal triangle the third entry is the sum of all the smaller counting numbers, namely the sum of $n - 1$, $n - 2$, and so forth, as far as 2, and finally 1. In symbols this sum is $(n - 1) + (n - 2) + (n - 3) + \cdots + 3 + 2 + 1$.
   (a) How many numbers are added together?
   (b) The sum can also be expressed by $1 +$ ____________________ (finish the expression by filling the blank, in accordance with the illustrative example).
   (c) Twice the sum is $n +$ ________________________________ (finish the expression).
   (d) How many numbers are added in part (c)?
   (e) What is their sum?
   (f) What is a simple name for the third entry in the n^{th} row of the Pascal triangle?

3. Fifteen points lie in a plane. No three of the points lie on

one line. How many lines are determined by these points?

*4. (a) What is the greatest common factor of the entries (excluding the 1's at the ends) in the seventh row of the Pascal triangle?

(b) Same question, for the fifth row.

(c) Same question, for the third row.

(d) What principle do you believe is illustrated in the preceding parts of this problem?

(e) Can you test your belief in later rows of the Pascal triangle? How?

(f) Perhaps your first guess concerning the principle is not verified by further testing. If this occurs in your case, can you modify your belief and find the correct principle?

*5. The dots in the following designs are arranged in equilateral triangular fashion.

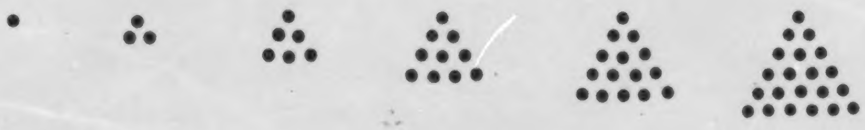
(a) How many dots are there in each design shown?

(b) How are the numbers in the answer to part (a) related to the Pascal triangle?

(c) If more designs of the same type and of successively larger size were drawn, how many dots would be in the eightieth diagram?

(d) How many dots in the thousandth diagram?

(e) In what sport is the fourth diagram used?



11-2. Permutations

Suppose that the club whose five members are Alice, Betty, Carol, Doris, and Ellen chooses an executive committee to conduct the business. The executive committee has three members and is composed of Betty, Doris, and Ellen. These three girls, in a meeting of the committee, decide that they should assign responsibilities. One should be chairman, another be secretary, and the third be treasurer for the club. In how many ways do you believe these jobs can be given to the three girls?

Class Discussion Exercises 11-2a

1. If Doris is chosen chairman, in how many different ways can the other two jobs be distributed between Betty and Ellen?
2. List each of these ways in detail, by telling which job each girl would have.
3. If Ellen is chosen chairman, in how many different ways can the other girls be given jobs?
4. How many different ways can the three offices be assigned to the three girls if Betty is chairman?
5. How many different ways can the three offices be assigned to the three girls?

Exercises 11-2a

1. A club has eight members whose initials are A,B,C,D,E,F,G,H. An executive committee {A,F,H} separates its jobs among its members. One possible way is:

   Chairman A, Secretary H, Treasurer F.

   (a) Make a list of all possible ways of assigning these three jobs to the three members of the committee so that each person has a job.

(b) How many ways are there?

(c) Does the factorization $3 \cdot 2$ give a clue concerning the why of the answer to part (b)?

2. The executive committee of a large club is $\{A, M, T, Y\}$. The committee decides to assign jobs to its members; the offices are president, secretary, treasurer, sergeant-at-arms.

   (a) If M is sergeant-at-arms, how many jobs and how many people are not yet matched?

   (b) If M is sergeant-at-arms, in how many ways can the other jobs be assigned?

   (c) If Y is sergeant-at-arms, in how many ways can the other jobs be assigned?

   (d) In how many ways can the officers be chosen if T is sergeant-at-arms?

   (e) How many possible choices are there for the sergeant-at-arms?

   (f) How many possible choices are there for giving each committee member one job?

   (g) Compare the answer to part (f) with four times the answer to Problem 1 (b).

   (h) Explain why the factor 4 appears in part (g).

   (i) In view of Problem 1 (c), does the factorization $4 \cdot 3 \cdot 2$ give a clue to the why of the answer to Problem 2 (f)?

3. Three boys, Ron, Sam, Ted, will participate, one after another in a contest.

(a) One possible order of performance is as follows: first Sam, second Ron, third Ted. Make a list of all the possible orders of performance.

(b) How many items are in your list?

(c) One way to make the list requested in part (a) is to fix attention on the positions and tabulate how the boys can be fitted in. Such a table might begin:

First	Second	Third
Sam	Ron	Ted
Sam	Ted	Ron

(and so on)

Can you finish the table to show all possibilities?

(d) Another way is to fix attention on the boys and tabulate which position will be assigned to each boy. Such a table might begin:

Ron	Sam	Ted
first	third	second
second	third	first

(and so on)

How would you finish this table?

(e) If you make such a list as is suggested in parts (a) or (c) or (d), do you understand why the answer to part (b) is the same as $3 \cdot 2$?

4. Many people are asked in a poll to express their preferences concerning potatoes: among the alternatives of baked potato, mashed potatoes, french fried potatoes, which do they like best, which next best, which least. How many different

orderings of preference are possible?

5. Four different presents are given to four children. In how many different ways can the gifts be distributed among the children?

6. A stenographer has four envelopes addressed to Adams, Brown, Clark, Davis respectively. She has four letters written to these four men. She puts one letter in each envelope. In how many ways can she do this so that one or more letters are placed in the wrong envelopes?

7. Five horses are to be assigned positions at the post in a race. In how many ways is it possible to distribute the horses among the first, second, third, fourth, fifth positions?

8. A salesman works five days during the week. He has customers in five cities. He spends one day each week in each city. Since he does not enjoy routine in his traveling, he likes to match the weekdays with the cities in as many ways as possible. How many weeks can he work without being obliged to repeat any day-city matching in his travels?

The matching of three girls in a club with three offices can be done in any of 6 ways. The matching of four letters with four envelopes can be done in any of 24 ways. The assignment of first, second, third, fourth, fifth positions at the post to five racers can be done in any of 120 ways. In each of these cases we have a set of objects (set of three girls, or set of five racers); we can arrange the members of the set in an ordered fashion (chairman, treasurer, secretary; or first, second, third, fourth, fifth). We inquire in how many ways the arranging can be done. Whenever we have a set which can be counted, an arrangement of its members in a certain order may be called a permutation of the set. In several recent problems we have been asking how many permutations a certain set has.

In the previous Exercises you studied several sets with three members and found that each of them has 6 permutations. Each of the sets with four members which you studied has 24 permutations. For five members the number of permutations is 120.

In the case of three members, you discovered that the number of permutations, six, has the factorization $6 = 3 \cdot 2$. For four members, we were interested in the factorization $24 = 4 \cdot 3 \cdot 2$. For five members, we observed that $120 = 5 \cdot 4 \cdot 3 \cdot 2$. Now $3 \cdot 2$, which is the same as $3 \cdot 2 \cdot 1$, is the product of all counting numbers as far as 3. The number 24, which is $4 \cdot 3 \cdot 2 \cdot 1$, is the product of all counting numbers as far as 4.

This type of product occurs so frequently in mathematics that a special description for it is convenient. If k is a counting number, the product of all the counting numbers up to and including k, is called the factorial of k. This product (in the United States) is commonly represented by the symbol $k!$ (pronounced, " k factorial").

As an example, $5!$ is the product of the counting numbers as far as 5; that is, $5!$ is the product of $1,2,3,4$, and 5; or $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. As another example, $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$. Check for yourself that $8! = 40320$. We agree that $1! = 1$.

Class Discussion Exercises 11-2b

1. Observe that $5! = (1 \cdot 2 \cdot 3 \cdot 4) \cdot 5$. Express $5!$ in terms of $4!$.

2. Show how 4! is related to 3!.
3. How is 8! related to 7!?
4. How is 436! related to 435!?
5. How is 8429 related to 8428? How is 8429! related to 8428!?
6. How is $k + 1$ related to k? How is $(k + 1)!$ related to $k!$?

We have seen that for certain sets with a few elements, the number of permutations of a set with k members is $k!$ Can you use the same type of approach and convince yourself of the following?

Property 1. If k is a counting number, then the number of permutations of any set with k members is $k!$.

Exercises 11-2b

1. Use the factorial notation to write $1 \cdot 2 \cdot 3$.
2. Find 6!.
3. Find the quotient of 14! by 13! (without performing any multiplications).
4. Show, without performing any multiplications, that 6! is the product of 6, 5, and 4!.
5. The factorial of 10 is the product of 10 and 9 and another factor. Give a new name for this third factor.
6. Show that 62! is the same as $62 \cdot 61 \cdot 60!$.
7. How many different batting orders are possible for a baseball team of nine players?
8. In a racing boat there are 8 seats, one behind another. In how many ways can the 8 members of a university crew take these seats?

9. How many permutations are there of the letters of the word "scholar"?

10. If one of the players on a baseball team always pitches, in how many ways can the players be distributed among the playing positions?

If you wish to buy a sundae at a certain dairy, you may choose flavors of ice cream and select the sauce. We may abbreviate the flavors by C, S, V (chocolate, strawberry, vanilla) and the toppings by M, N (marshmallow, nut). A marshmallow sundae on chocolate ice cream may be described by the pair (C,M).

Class Discussion Exercises 11-2c

1. What kind of sundae is described by the pair (V,N)?
2. Make a list of all the possible pairs.
3. How many pairs are there?
4. How many choices are there for the left member of a pair?
5. How many choices are there for the right member of a pair?
6. What is the relationship among the answers to Questions 3,4,5?
7. If there were five flavors of ice cream instead of three, how many kinds of sundaes would be possible?
8. If there were four flavors of ice cream and three kinds of syrup, how many kinds of sundaes would be possible?

Suppose there are four packages in a grab-bag; we may call them A, B, C, D. You are to choose one, give it to your friend, choose another, keep it for yourself. What possibilities are there? We may use the pair (A,C) to represent the possibility that package A is chosen for your friend and package C for you. The symbol (C,A) indicates that your friend is given C and you keep A.

9. What does the symbol (D,A) mean?
10. Explain why (B,D) and (D,B) are different in meaning.
11. Why does the symbol (C,C) have no meaning at all in this discussion?
12. Make a list of all the possible pairs.
13. How many pairs are there?
14. How many choices are there for the left member of a pair?
15. How many choices are there for the right member of a pair?
16. If the left member of a pair is known, how many choices are there for the right member of the pair?
17. Does the answer to Question 16 depend on which package is represented by the left member of the pair?
18. What is the relationship among the answers to Questions 13, 14, 16?
19. Is the relationship among the answers to Questions 13, 14, 16 a much simpler one than any relationship among the answers to Questions 13, 14, 15?

Our last two illustrations provide examples of Property 2, which we will soon state. In both illustrations we found that the total number of pairs can be computed as a product. When there are many pairs, this method of finding how many is often better than counting the pairs individually. In each illustration, one factor of the product is the number of possible left members in the various pairs. The other factor tells the number of pairs which have a given left member.

In general, if the number of possible left members is r and if each of these occurs in s pairs, then the total number of pairs is $r \cdot s$. In the example concerning sundaes, $r = 3$ and $s = 2$, and the number of kinds of sundaes is $r \cdot s = 3 \cdot 2 = 6$. In the grab-bag example, $r = 4$ and $s = 3$, and the number of pairs is $4 \cdot 3 = 12$. We now state our result in words.

Property 2. Suppose that in a set of pairs each possible left member appears in the same number of pairs as every other left member. Then the total number of pairs is the product of the number of possible left members and the number of pairs which have a given left member.

Exercises 11-2c

1. A boy has seven shirts and four pairs of trousers. How many different costumes does he have?
2. A baseball team has five pitchers and three catchers. How many batteries (consisting of a pitcher and a catcher) are possible?
3. If the first two call letters of a television station must be KT, how many different calls of four letters are possible?
4. A disc jockey has 50 records in his collection. He wants to make a program of two different songs. How many possible programs are there?
5. A signalman has six flags. The emblems on the various flags are a stripe, a dot, a triangle, a rectangle, a bar, and a circle. By showing two different flags, one after the other, the signalman can send a signal. How many different signals are possible?

*6. Consider the set of nine numbers 1,2,3,4,5,6,7,8,9.

   (a) How many (ordered) pairs of different numbers chosen from this set are there?

   (b) How many (ordered) pairs of numbers chosen from this set have the property that the members of the pair differ by two or more?

   (c) How many (ordered) pairs of these numbers have the property that the right member of the pair is less than the left member?

   (d) Answer each of the parts (a), (b), (c) in the case where the given set of numbers includes all the whole numbers from 0 to 27 inclusive.

We often extend Property 2 to triples. A set of five flags has one each of the colors, black, green, red, white, yellow; we use the symbols B,G,R,W,Y. Three flags are to be hoisted, one above another, on the same mast. Each arrangement is a signal to a distant observer. The triple (G,Y,B) means that the green flag is at the top of the pole, the yellow flag in the middle, and the black flag is the lowest.

Class Discussion Exercises 11-2d

1. Explain the distinction among the triples (B,W,R), (R,W,B), and (W,R,B).
2. How many choices (from the five initial letters representing flags) are there for the left member of a triple?
3. How many choices are there for the middle member of a triple?
4. If the left member is known, how many choices are there for

the middle member of the triple?

5. Use Property 2 to find how many choices there are for the top pair of positions on the flagpole.
6. How many choices are there for the right member of the triple?
7. If the first two members of a triple are known, how many choices are there for the right member of the triple?
8. Does the answer to Question 7 depend upon which two flags are represented by the first two members of the triple?
9. Can you use Property 2 again to find how many possible triples there are?
10. How many different signals can be sent by hoisting three flags as described?

The example we have just studied illustrates the notion of the number of permutations of a set of 5 objects taken 3 at a time. Let n and r be counting numbers such that $r < n$. If a set of n objects is given, a selection of r of the objects together with an arrangement of those r objects in an ordered fashion is called a permutation of the n objects taken r at a time.

The steps in the last illustration serve as a model for us in developing a formula for the number of permutations of n objects taken r at a time. As you study the next two paragraphs, keep in mind the answer to Question 9 in the last illustration. The answer is the same as $5 \cdot 4 \cdot 3$; the first factor is five, five is the number of objects (flags), each succeeding factor is one less than the factor before, the number of factors is three, three is the number of objects taken (number of flags in a triple). Now turn to the general situation in which we arrange r objects chosen from n objects altogether.

There are n possible choices for the first position in the ordering. For each such choice there remain $n - 1$ objects not yet selected, and consequently there are, at this stage, $n - 1$ possible choices for the second position in the ordering. According to Property 2, the number of choices for the first two positions together is $n \cdot (n - 1)$. For each of these, there are $n - 2$ possibilities for the third position. According to Property 2, there are $n(n - 1) \cdot (n - 2)$ possibilities for the first three positions together.

We continue in this fashion until the final stage. At that stage, $r - 1$ objects have been selected (Why?), $n - r + 1$ objects remain not yet selected (Why?). The number of permutations of n objects taken r at a time is $n(n - 1)(n - 2)\cdots(n - r + 1)$.

As an illustration, in the discussion of the flag triples, $n = 5$ and $r = 3$, consequently $n - r + 1 = 5 - 3 + 1 = 3$ and $n(n - 1)(n - 2)\cdots(n - r + 1) = 5 \cdot 4 \cdot 3 = 60$.

Note that the symbol $n(n - 1)(n - 2)\cdots(n - r + 1)$ represents the product of all the counting numbers between $n - r + 1$ and n inclusively. Thus when $n = 5$ and $n - r + 1 = 3$, the factors of the product are all the counting numbers between 3 and 5 inclusive; that is, the factors are 3,4,5.

The number of permutations of a set of n objects taken r at a time is frequently indicated by the symbol $P_{n,r}$. Using this notation, we have just found that $P_{5,3} = 60$. Check for yourself that $P_{6,2} = 30$.

Exercises 11-2d

1. Find $P_{4,2}$.
2. Write the symbol for the number of permutations of 27 objects taken 19 at a time.
3. Write the number $P_{20,6}$ in expanded form (but do not multiply).
4. Write the new symbol which represents the product $43 \cdot 44 \cdot 45 \cdot 46 \cdot 47 \cdot 48 \cdot 49$.
5. (a) Write the number $P_{12,3}$ in expanded form (but do not multiply).

   (b) Write the product of $P_{12,3}$ and $9!$ in expanded form (but do not multiply).

   (c) What convenient name do we have for the product in part (b)?

   (d) In part (b), how can the numbers 12 and 3 be combined to yield 9?

   (e) Express $P_{12,3}$ as the quotient of two numbers, each of which is a factorial.
6. (a) Write the number $P_{20,4}$ in expanded form (but do not multiply).

   (b) Write the product of $P_{20,4}$ and $16!$ in expanded form (but do not multiply).

   (c) What convenient name do we have for the product in part (b)?

   (d) In part (b), how can the numbers 20 and 4 be combined to yield 16?

   (e) Express $P_{20,4}$ as the quotient of two numbers, each of which is a factorial.

*7. (a) Write the number $P_{n,r}$ as a product.

(b) Use the clues in Problems 5(d) and 6(d) and select a certain factorial.

(c) What is the product of $P_{n,r}$ and the number you chose in part (b)? (Use the clues in Problems 5(c) and 6(c), if necessary.)

(d) Express $P_{n,r}$ as a quotient of two numbers, each of which is a factorial.

8. (a) A man has a combination lock with 50 numbers on the dial. He has forgotten the combination but he remembers that the first turn is toward the right and that three different numbers (in a particular order) are required to open the lock. How many possible trials may be necessary to open the lock?

(b) A man has a combination lock with 50 numbers on the dial. He has forgotten everything about the combination except that there are three numbers (in order) required. How many possible trials may be necessary to open the lock?

9. A telephone dial has a finger hole for each of the ten digits.

(a) How many telephone numbers, each with five digits, are possible?

(b) How many telephone numbers, each with five digits but with no digit repeated, are possible?

10. Five players on a football squad can play either left end or right end. In how many ways may the coach choose the two ends for the opening lineup?

*11. Suppose we want to send messages in a code. We use certain symbols, say n of them. (The symbols might be letters or flags or sounds or designs or any other type of symbol.) Each message is composed of four different symbols, arranged in order. The number of possible messages which we may wish to send is 1600. What is the smallest number that n can be in order to meet the requirements?

11-3. Combinations

Whenever we have been using the word "permutation", we have been concerned, not only with the elements, but also with the arrangement or the ordering of the elements. At the beginning of this unit we discussed committees in a club. In a committee such as we studied, the members are not arranged or ordered in any manner. The choosing of a committee from a club is an illustration of a combination. A combination of a certain set of n objects taken r at a time is a selection of r members from the set with no regard to ordering the chosen members. We are here letting n be a counting number and r be a whole number no greater than n. The number of combinations of a set of n objects taken r at a time is often represented by the symbol $\binom{n}{r}$. In this unit you may read this symbol by saying: "the number of combinations of n things r at a time"; sometimes we may pronounce it simply "n, r"; in a later unit you may find another way in which the symbol can be read.

The entries in the Pascal triangle are values of $\binom{n}{r}$. For example, from the fifth row of the pascal triangle we find (reading from the left) that

$\binom{5}{0} = 1$, $\binom{5}{1} = 5$, $\binom{5}{2} = 10$, $\binom{5}{3} = 10$, and so on.

You will want to note that the new symbol we have introduced can be easily distinguished from a fractional symbol, because the new symbol has no bar between the two numbers.

Exercises 11-3a

1. Write the special symbol for each of the following:
   - (a) the number of combinations of 12 objects taken 7 at a time;
   - (b) the number of permutations of 12 objects taken 7 at a time;
   - (c) the number of combinations of m things taken 3 at a time;
   - (d) the number of combinations of n + 2 objects taken k at a time.
2. Read each of the following symbols: $\binom{6}{2}$; $P_{8,4}$; 52!; $\binom{52}{13}$; $P_{9,7}$; $\frac{8}{4}$.
3. Use the Pascal triangle to find each of the following:
   - (a) $\binom{6}{2}$ and $\binom{6}{4}$;
   - (b) $\binom{7}{4}$ and $\binom{7}{3}$;
   - (c) $\binom{8}{3}$ and $\binom{8}{5}$;
   - (d) $\binom{3}{1}$ and $\binom{3}{2}$;
   - (e) $\binom{8}{2}$ and $\binom{8}{6}$.

*4. (a) Suppose that a and b are two counting numbers and let s be the sum $a + b$. Does Problem 3 give you a clue concerning an important relationship between $\binom{s}{a}$ and $\binom{s}{b}$?

(b) If so, what is the relationship?

(c) Can you use some ideas in Section 1 to convince yourself that this relationship is true in every case?

5. (a) Find each: $\binom{4}{4}$, $\binom{6}{6}$, $\binom{9}{9}$, $\binom{1247}{1247}$.

(b) What general notion do these examples illustrate?

6. (a) Find each: $\binom{4}{0}$, $\binom{6}{0}$, $\binom{7}{0}$, $\binom{249}{0}$.

(b) What general notion do these examples illustrate?

7. Show that if n is a counting number different from one, then $\binom{n}{2} = \frac{n(n-1)}{2}$.

Suppose that a club of seven members picks three officers. With the aid of the Pascal triangle, we learned that the number of these possible executive committees is 35. This number 35 we may now call $\binom{7}{3}$. By Property 1 we learned that the three offices may be matched with the three officers in 6 ways. This number 6 we observed is $3!$. We may apply Property 2 and find that $35 \cdot 6$ is the number of possible officer assignments. This number $35 \cdot 6$ is $\binom{7}{3}(3!)$.

On the other hand we may apply Property 2 (extended) to find that the number of choices of three members, arranged by office, from the club of seven members is $7 \cdot 6 \cdot 5$, namely 210. This number 210 is $P_{7,3}$.

Both viewpoints yield the same count, of course. We see that

$\binom{7}{3}$ $(3!) = P_{7,3}$. The same type of argument shows that, for any two counting numbers r and n, if $r < n$, then

$$P_{n,r} = \binom{n}{r} \ (r!).$$

Consequently,

$$\binom{n}{r} = \frac{P_{n,r}}{r!} .$$

Since $P_{n,r} = n(n - 1)\cdots(n - r + 1)$, we obtain

$$\binom{n}{r} = \frac{n(n - 1)(n - 2)\cdots(n - r + 1)}{r!}.$$

In this fraction, the number of factors in the numerator is r, the same as the number of factors in the denominator.

Exercises 11-3b

1. Ten men are qualified to run a machine that requires three operators at a time. How many different crews of three are possible?
2. A disc jockey has a set of 15 records. Each night he selects 5 records to make a program. How many nights can he do this without repeating any program? Disregard the order in which the individual records occur within a program. (You do not need to perform any multiplications, but may leave your answer in whatever symbols you think are convenient.)
3. Eight points are given in space, and no four of them lie in the same plane. (Remember that any three of them determine a plane.) How many different planes are determined by the eight points?
4. On a certain railway there are 12 stations. How many different kinds of tickets should be printed to provide tickets between

any two stations,

(a) in case the same ticket is good in either direction;

(b) in case different tickets are needed for each direction?

5. A man has six bills, one each of the amounts \$1, \$5, \$10, \$20, \$50, \$100. How many different sums of money may be found by using one or more of these six bills together?

6. A restaurant has prepared 4 kinds of meat, 3 kinds of salad, and 5 kinds of vegetable. A platter consists of a meat, a salad, and a vegetable. How many different kinds of platters are possible?

7. There are eight teams in a baseball league. During the season each team plays every other team five times. How many games are played in the league altogether during one season?

8. Nine boys and eight girls attend a party. Each dance there are eight couples on the floor and one boy sitting out; each dance lasts five minutes. Suppose that the party continues until every possible matching of the boys and girls in a dance has occurred. How many hours has the party lasted?

9. Either one or two of a string of eight Christmas tree lights wired in series are burned out. Suppose you have two good bulbs and suppose you try, first one at a time, then two at a time, to locate the burned out bulb (or bulbs). How many trials might it be necessary for you to make in order to find the bulb (or bulbs) that need replacement?

10. A girl has four skirts, six blouses, and three pairs of shoes. How many weeks can pass while she wears a different costume every day?

11. In the game of bridge, a hand consists of 13 cards from the playing deck of 52 cards. The number of possible bridge hands is 635,013,559,600. Write this number, using a special symbol you have studied in this unit.

12. A salesman has customers in eight cities away from home. He wishes to plan a travel route which will take him to each of the eight cities in turn and afterwards back to his home. How many possible routes are there?

*13. Use Problem 7(d) from Exercises 11-2d and show that

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}\ .$$

UNIT 12

PROBABILITY

12-1. Chance Events

This unit will deal with chance events, that is, events which may occur when experiments are performed whose outcomes are uncertain. For example, a weatherman makes a forecast of the future weather, but he cannot influence the forces of nature. His forecast, "Rain" is more accurately a probability statement, "It will probably rain." Similarly, you may predict that "The Pink Shirts will win the pennant," but what you mean to say is "It is likely that the Pink Shirts will win the pennant." An honest gambler may know the "odds" and use this knowledge to win, but he cannot resort to magic tricks by influencing the outcome and remain honest.

Some examples of games of chance will be used to help you understand what probability means. Such games give us excellent mathematical models for use in studying the laws of probability. The examples are not used with the idea that gambling is to be encouraged. Rather, the information in this unit should help you begin to understand why "most gamblers die broke."

Probability has many practical uses: federal and state governments use it in setting up budget requirements; military experts use it in making decisions on defense tactics; scientists use it in research and study; insurance companies use it in setting up life expectance tables; weather forecasting was mentioned earlier; and there are other uses too numerous to mention here.

In Section 1 we shall study some ideas about statements involving chance events, like "The Green Sox will win," or "Sandlot will win the race," or "If I toss a coin and allow it to fall freely, it will show heads." A statement is a collection of words or symbols which is either true or false (but naturally not both.) If a statement which predicts a future event turns out to be correct, we will call it a "true statement." When it does not turn out to be correct, we will call it a "false statement." In this section we will concern ourselves with the "measure of chance" that a statement predicting a future event is true. This measure of chance is also called the probability of the statement.

The measure of chance, or probability, is sometimes easier to understand if we use a mathematical model where we can count the possible outcomes. For example, the game of tossing a coin, commonly called "heads or tails" can be used as a model. If we toss a coin and allow it to fall freely, either a head will show or a tail will show. We assume the coin is not weighted. Such a perfectly balanced coin is sometimes called an "honest coin."

Consider the statement,

"If we toss a coin and allow it to fall freely it will show heads."

The table below shows the number of possible outcomes:

Possible Outcomes	Truth Value of the Statement
Heads	T
Tails	F

If the outcome is heads, then the statement is true, and we have indicated this by writing "T" in the truth value column. On the other hand, if the outcome is tails, then the statement is false, and its truth value is represented by the initial letter "F".

The table above is called a "truth table." A truth table for a statement lists all the possible outcomes of the experiment referred to by the statement. Since our coin is balanced, we believe that, in this situation, the chance of the statement being true is one out of two.

In probability it is useful to use a number to indicate the measure of chance that a statement is true. The measure of chance for the above statement is written as $\frac{1}{2}$.

If an event is governed by chance, then it has a certain probability of happening. This probability will be a number between 0 and 1. If we use the letter "A" to represent the event of the coin showing heads, then we can call $\frac{1}{2}$ the probability of the event A . This is the same as saying that the measure of chance that the above statement is true is $\frac{1}{2}$. We can represent the probability of the event A as

$$P(A) = \frac{1}{2}$$

If we use the latter "B" to represent the event of the coin showing tails, we are concerned with the probability of B . We can represent this with the symbol $P(B)$. Thus,

$$P(B) = \frac{1}{2}$$

It is important that you understand that in the above case $P(A) = P(B)$. That is, each event is equally likely to occur. Any two statements which predict events that are equally likely have the same probability.

Suppose you have tossed an honest coin five times and it shows heads each time. What is the probability that the coin will show tails on the next toss? Some people believe that the odds will change, that the forces of "luck" will act to force the coin to show tails until a balance of sorts is restored between the showing of heads and tails. Not so! The probability that the coin will show heads remains $\frac{1}{2}$ for each toss. In probability we do not say that if the coin shows heads on the first toss it must show tails on the second toss. We cannot predict whether the coin will show heads or tails.

Suppose you use two pennies. What is the measure of chance that the statement "If two coins are tossed, one head and one tail will show" is true? That is, what is the probability that the event of one head and one tail showing will occur. The table below shows that there are four likely outcomes:

Possible Outcomes		"x" indicates that the event has occurred, "o" that it did not.
First Coin	Second Coin	
Heads	Heads	o
Heads	Tails	x
Tails	Heads	x
Tails	Tails	o

Note that there are two outcomes showing one head and one tail. The probability that the event will occur is $\frac{2}{4}$ or $\frac{1}{2}$. If we use the letter "E" to represent the event, we may write the following:

$$P(E) = \frac{1}{2}$$

What is the probability that exactly two heads will show if two coins are tossed. Check the truth table above. Of the four outcomes, how many ways are there for this event to occur? If we use the letter "G" to represent the event, we may write the probability of the event G as

$$P(G) = \frac{1}{4}$$

Note that in this example events E and G are not equally likely. Their probabilities are different.

When we find the probability of an event we shall say that we are estimating the probability from tbe best information we have. We may summarize our results by a formula:

$$P(E) = \frac{t}{s}$$

Where P represents the probability that an event E will occur
t is the number of ways in which E can occur,
s is the total number of possible outcomes.

Thus this is a counting procedure. Later you will learn that more advanced mathematical processes have been set up to find probability where counting is impossible. However, counting is essential in many situations.

Exercise 12-1

1. Two black marbles and one white marble are in a box. You are to take out, without looking inside the box, two marbles. Find the probability of the event that when, without looking, two marbles are taken out of the box, one marble is black and the other is white.

   Suggestion. Complete the table below. List all possible outcomes. Indicate which outcome shows the event occurring with an "X" and those not occurring with an "O".

Possible Outcomes		Occurrence of the Event	
B_1	B_2	O	
B_1	W	?	P = ?
?	?	?	

2. Using the data in problem 1, find P for the event that if two marbles are taken out of the box, both marbles will be black.
3. If three honest coins are tossed, what is the chance that three heads will show? (Sometimes we say "What is the chance?" when we mean "What is the measure of chance that the statement is true?") Make a table showing all possible outcomes.
4. Using data in problem 3, find P for the event that if three honest coins are tossed exactly one tail and two heads will show. (Note that the number of events that can occur in this case is different from the number of events occurring in Problem 3.)
5. Three hats are in a dark closet Two belong to you and the other to your friend. Being a polite person, when your friend is ready to leave with you, you reach in the closet and draw any two hats. What is the probability that you will pick the two wanted hats?
6. A full deck of playing cards contains 52 cards. There are four different suits, or groups, of cards: hearts and diamonds are colored red; spades and clubs are colored black. Each suit contains 13 cards, one-fourth of the total number of cards. Each suit contains cards numbered from one to ten inclusively and each suit also contains a jack, a queen, and a king.

Find P for the event that if a card is drawn from a full deck of playing cards, it will be a spade.

7. The cards marked as "ones" are called "aces." What is the chance of picking an ace from a full deck of cards?
8. Find P for the following event; if a card is drawn from a deck it will be the ace of spades.
9. (a) If a joker (extra card) is placed in a full deck of playing cards, what is the chance of drawing a queen from the deck in a single draw?
   (b) Why are the measures of chance for problem 7 and 9(a) different?
10. Suppose a box contains 48 marbles. Eight of the marbles are black and forty of the marbles are white.
    (a) How many marbles are there altogether?
    (b) How many white marbles are there?
    (c) Find P for the event that if a marble is picked at random (without looking in the box), it will be white. Reduce the fraction to lowest terms.
11. In problem 10, what is the chance that the first marble picked will be black?
12. Using the data for problem 1, consider the event "If, without looking, two marbles are taken out of the box, both marbles will be white."
    (a) Is the outcome in this case possible?
    (b) What measure of chance can we assign to such an outcome?
13. Using the date in problem 3, find P for each of the following:
    (a) If three honest coins are tossed, three heads will show.
    (b) If three honest coins are tossed, three tails will show.
    (c) If three honest coins are tossed, two heads and one tail will show.
    (d) If three honest coins are tossed, two tails and one head will show.
    (e) Does $P = 0$ for any of the above?
    (f) Does $P = 1$ for any of the above?

(g) Find the sum of the probabilities for (a) through (d) above. You should get "1" . Why?

*14. Suppose you have five cards, the ten, jack, queen, king, and ace of hearts. As you draw a card from the group you lay it aside. You do not replace the cards after each draw.

(a) What is the chance that the first card you darw is the ace?

(b) Assume you draw the jack on the first draw. What is the chance that the second card you draw is the ace?

(c) Are your answers for (a) and (b) the same? Why?

(d) After drawing the jack, assume the second card you draw is the ten. What is the chance that the third card you draw is the ace?

(e) What is happening to the measure of chance as the number of cards decreases?

*15. "He has a 50 - 50 chance of winning the election."

(a) What is the probability that he will win?

(b) Suppose a measure of chance is less than $\frac{1}{2}$. What does this mean in terms of the outcome of an event? Is the outcome very likely or not very likely to occur?

*16. Suppose you have tossed an honest coin nine times and it shows heads each time.

(a) Consider the above as one event. Is this event likely to occur? Explain your answer.

(b) What is the probability that the coin will show tails on the tenth toss?

(c) Does the outcome of the first 9 tosses have any affect on the tenth toss?

12-2. Probability for Models

In the previous exercise you determined the measure of chance, which we call probability, by listing all possible outcomes, as in a truth table. This is easy when there are only one or two coins, but as the number of coins increases, it is difficult to remember all the possibilities or to avoid listing some possibilities twice. Let us see if we can discover an easy, accurate way to make these listings.

The table for two coins shows this pattern: ("H" represents heads and "T" represents tails.)

Possible Outcomes

First Coin	Second Coin
H	H
H	T
T	H
T	T

Note that the first column is grouped by twos; HH , TT . The second column is grouped alternately; H, T , H , T . Compare the pattern in the table for two coins with the pattern in the table for three coins shown below:

Possible Outcomes

First Coin	Second Coin	Third Coin
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T

Note how each column is grouped: the first by fours; the second by twos; the third alternately.

How many possibilities are there in the table for two coins? Note, there are four. How many possibilities are there in the table for three coins? There are eight. Can you find a relation between the number of sides (two) of the coin, and the number of possible outcomes? With two coins each having two sides there are four possible outcomes. With three coins each having two sides there are eight possible outcomes. Can you predict the number of possible outcomes with four coins?

If we use the number of sides, 2 , as the base, and the number of coins, 2 or 3 and so on, as the exponent, we can determine the number of possible outcomes without listing them in table form. We need only determine the number represented in exponential notation. If we use two coins there will be . . .

2^2 or 4 possible outcomes.

If we use three coins there will be . . .

2^3 or 8 possible outcomes.

We can express this result as a formula:

$$T = S^n$$

Where T is the total number of possibilities,

S is the number of sides of the coin or object used (a cube or some other object with more than two faces may also be used),

n is the total number of coins or objects used.

Having determined the total number of possible outcomes, it is then easy to set up the table. Recall the patterns in the table for two coins and the table for three coins. If we use 4 coins, how many outcomes must be listed in a table? By using the formula $T = S^n$ where $S = 2$ and $n = 4$, we find

$$T = 2^4$$
$$T = 2 \cdot 2 \cdot 2 \cdot 2$$
$$T = 16 .$$

There are 16 possible outcomes with four coins. What will the pattern be for the first column? The second column? The third column? The fourth column?

In making truth tables, such as those in section 1 and those described above, experimental observations were not used. Rather, all possible outcomes of events were listed, as in the cases of tossing two or three coins. The probability is based on the outcomes listed in the table, or model as it is sometimes called. In the tables previously discussed we assume that each separate possibility, or outcome, has the same chance of occurring. We say that each outcome is "equally likely to occur."

Sometimes we find it necessary to make observations and list the possible outcomes. That is, we try to find the probability that an event will occur when we cannot list all possible outcomes. For example, the table below shows a small number of weather forecasts, only those from April 1 to April 10. The actual weather is also shown.

Table

Date	Forecast	Actual weather	"X" indicates the event did occur, "O" that it did not
1.	Rain	Rain	X
2.	Light showers	Sunny	O
3.	Cloudy	Cloudy	X
4.	Clear	Clear	X
5.	Scattered showers	Warm and sunny	O
6.	Scattered showers	Scattered showers	X
7.	Windy and cloudy	Overcast and windy	X
8.	Thundershowers	Thundershowers	X
9.	Clear	Cloudy and rain	O
10.	Clear	Clear	X

Observe that forecasts 1, 3, 4, 6, 7, 8, and 10 were correct. We have observed ten outcomes. The event of a correct forecast has occurred seven times. Based on this information, the probability that future forecasts will be true is $\frac{7}{10}$. This number is the best estimate that we can make from the given infromation. In this case, since we have observed such a small number of outcomes, it would not be correct to say that our estimate of P is very accurate. A great many more cases should be used if we expect to make a good estimate of P. Note that the table above is similar to tables used in section 1, but in the above table we listed only ten outcomes, not all possible outcomes.

A physicist cannot trace the motion of a single molecule of oxygen in a room, but he can estimate the probability that an oxygen molecule will hit one of the walls in a room in the next second. To make such conclusions requires an understand of much more mathematics than we can study in this unit.

While probability has many uses in games such as those mentioned in section 1, the theory of probability is very important whenever we deal with mass events as the one mentioned in the previous paragraph. Most events are compound events that are made up of many single events:

1. The occurrence of 300 traffic deaths in the U. S. in one day is a mass event.
2. The occurrence of 20 absences in your school on Monday is a mass event. This is made up of many simple events such as: Terry came to school, Susan came to school, Mark stayed home, and so on for all the students in the school.

Section 1 dealt with some simple events governed by chance. We assigned measures of chance, which we called probabilities, for the outcomes of these events. The numbers we used to represent "P" were numbers like one-half, one-third, one-fourth, and so on. When we used a table to conclude that $P(A) = \frac{1}{2}$, we were writing an "estimate" of the probability. However, if we actually toss an honest penny once, we cannot predict whether it will show heads or tails. But if we toss an honest penny a million times, then it is almost certain that the number of heads will be between 490,000 and 510,000. The ratio of heads that show to the number of tails that show will be almost certainly between $\frac{49}{100}$ and $\frac{51}{100}$. We cannot in this unit study all the mathematics required to make such conclusions. We can get a clue, however, of how to find the probability that an event will occur from the information listed in a table.

Exercise 12-2

1. A truth table shows all the possible outcomes for tossing 5 coins. Without listing them, determine the number of possible outcomes.

2. A regular tetrahedron is a solid having four faces. The letters A, B, C, and D are printed on the faces.
    - (a) If a regular tetrahedron is rolled, how many possible ways are there for it to stop? (Note that in this case we will consider the side on which the object rests as an outcome. That is, the face that is the base may be marked A, B, C, or D.)
    - (b) How many outcomes are there if two such tetrahedrons are rolled?
    - (c) How many outcomes are there if three such tetrahedrons are rolled?
    - (d) Find the measure of chance for the following statement: "If the tetrahedron is rolled it will stop on side A."
3. The letters A, B, C, D, E, and F are printed on the side of a cube.
    - (a) If one cube is rolled, how many possible outcomes are there? (We will consider the side facing up as the outcome in this case.)
    - (b) If two cubes are rolled at the same time, how many outcomes are there?
    - (c) What is the chance that B will show if one cube is rolled?
    - (d) What is the chance that two E's will show if two cubes are rolled at the same time?
4. (a) How many likely events are possible if a regular octahedron (an eight sided solid) is rolled?
    - (b) How many outcomes are possible if two regular octahedrons are rolled simultaneously (at the same time)?
5. Consider the following events:

    A. It rains on Friday, the 13th.

    B. The sun shines on Friday, the 13th.

    The following table shows the weather on 20 Friday, the 13ths. Using the information listed in the table, find P for the events A and B. Based on the information in the table, which is more likely to occur over a great number of Friday, the 13ths, A or B ? The first part is completed. Complete the table.

"X" indicates the event did occur.
"O" indicates the event did not.

Weather on 20 Friday, the 13ths		A	B
1.	Heavy rain	X	O
2.	Light rain	X	O
3.	Sunny	O	X
4.	Usually sunny, no rain	?	?
5.	Sunny	?	?
6.	Scattered showers	?	?
7.	Showers	?	?
8.	Sunny	?	?
9.	Sunny	?	?
10.	Sunny	?	?
11.	Cloudy, no rain	?	?
12.	Partly cloudy	?	?
13.	Cloudy with some showers	?	?
14.	Showers	?	?
15.	Sunny	?	?
16.	Sunny	?	?
17.	Hot and sunny	?	?
18.	Sunny	?	?
19.	Cloudy and some showers	?	?
20.	Sunny	?	?

(Note: it is possible that neither event occurred.)

6. There are 35 pieces of brick of which 5 are gold. What is the chance that if you pick a brick at random you will pick a gold one? (At random means "without looking".)

7. (a) If one penny is tossed what is the chance that a head shows?
   (b) How many heads would you expect to get if the penny is tossed 50 times?

8. An urn contains 5 white balls, 3 black balls, and 2 gray balls. (An urn is a container like a jar or bowl.)
   (a) What is the chance that you will pick a white ball in one draw?
   Assuming you pick a white ball the first time and did not

replace it, what is the chance that you will pick a black ball the second time?

(c) Assuming you pick a white ball the first time and a black ball the second time and did not replace them, what is the chance that you will pick a gray ball the third time?

9. (a) What is the chance of picking the 9 of spades from a deck of cards?

(b) What is the chance that some 9 will be picked from a deck of cards?

10. Suppose on a regular dodecahedron, a solid having twelve plane faces, 5 faces are colored white and 7 faces colored black. If you toss it, what is the chance that it will stop with a white side down?

*11. Suppose you have six letters to be delivered in different parts of town. Two boys offer to deliver them. In how many different ways can you distribute the letters to the boys?

*12. Three cards are numbered like the ones below.

$\boxed{3} \quad \boxed{5} \quad \boxed{6}$

Without looking at the numbers you are to draw any two cards. What is the chance that the sum of the two numbers is odd?

*13. You are to be placed in a line with two girls (or boys), one of whom is your favorite. What is the probability that you will stand next to your favorite? In such a problem we assume that you are not placed according to any plan (including your own). If you crowd in next to your favorite, chance would not play a role.

*14. If there are 225 white marbles and 500 black marbles, what is the chance of picking a black marble on the first draw?

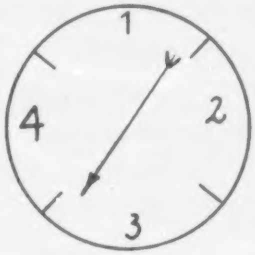
*15. When six coins are tossed, what is the chance that one and only one should turn up a head?

12-3. Mutually Exclusive Events

In mathematics we are always looking for general principles which describe a certain situation. In this section and the next we will identify two of the most important general principles of probability.

Consider the following problem:

A dial and a pointer like the one illustrated will be used for the problem. The pointer spins and we can tell whether it stops at 1, 2, 3 or 4. What is the probability that the pointer will stop at an even number?



There are four possible outcomes. The pointer can stop at 1, 2, 3 or 4. The event in question occurs if it stops at 2 or 4, that is, the event occurs in two out of the four possible outcomes. Thus the probability of the hand stopping at an even number is $\frac{1}{2}$.

This event is a combination of two other events. Let A be the event of the pointer stopping at 2, and B be the event of the pointer stopping at 4. If we use the symbol "A or B" to stand for the event, either event A or event B occurs, then "A or B" is the event of the pointer stopping at an even number. We have found that

$$P(A \text{ or } B) = \frac{1}{2} .$$

Could we find this probability by considerinv events A and B individually? We know that

$$P(A) = \frac{1}{4} \quad \text{(Why?)}$$

and

$$P(B) = \frac{1}{4} \quad \text{(Why?)}.$$

If we add $\frac{1}{4}$ and $\frac{1}{4}$, the result is $\frac{1}{2}$. How can we obtain P(A or B) from P(A) and P(B) ?

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$$

$$P(A \text{ or } B) = P(A) + P(B)$$

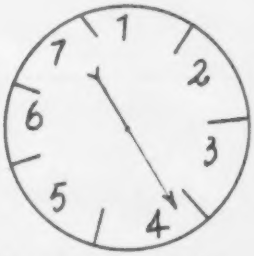
Many times (as in the above case) we can add probabilities of individual events to find the probability of another event. Notice that in the case above, the pointer could not stop at 2 and 4 at the same time (as a result of one spin). For one spin it had to stop at one or the other. Events A and B could not both occur at once. This is one of the conditions that must be met before we can add probabilities. Two events which cannot occur at once are called mutually exclusive events.

Property: If A and B are two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B) .$$

This proposition holds also when we have 3 or more mutually exclusive events.

Example: Use the definition to answer the following:



The seven numbers are equally spaced.

The hand spins freely. What is the probability that it will stop at an even number?

There are 3 single events: A, stops at 2; B, stops at 4; C, stops at 6. The event whose probability we seek is A or B or C . Events A, B and C are mutually exclusive, since the hand can stop at only one of the numbers as the result of one spin. Therefore,

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) .$$

$P(A) = 1/7$ (Why?) . Also $P(B) = 1/7$ and $P(C) = 1/7$.

Thus $$P(A \text{ or } B \text{ or } C) = 1/7 + 1/7 + 1/7 = 3/7 .$$

Exercises 12-3

1. Mutually exclusive events then are events which cannot happen at the same time. If one event happens, the other cannot. With this in mind, which of the following events are mutually exclusive:

   (a) The event of throwing a head or a tail on a single toss of a coin.

(b) The event of your mother making breakfast or your going to school.

(c) The event of throwing a head or of rolling a 3 on a die.

(d) The event of rolling a six sr a three on a cube with faces numbered 1 to 6.

(e) The event of driving the car or going to the store.

(f) The event of going upstairs or going downstairs.

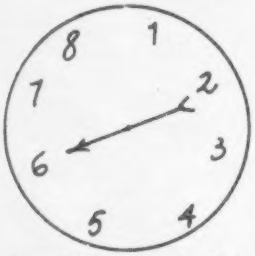
(g) The event of drawing an ace or a jack from a deck of cards on a single draw.

(h) The event of running or sitting.

(i) The event of talking to your teacher or of talking to your mother.

(j) The event of stopping the car and of starting the car.

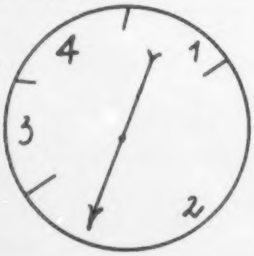
2. Use the formula to solve the following problem.



The numbers are equally spaced around the dial.

What is the probability that the spinning hand will stop at an odd number?

*3.



2 is a semi-circle
1, 3 and 4 are equally spaced

Use the formula to solve the problem:
What is the probability that the spinning pointer will stop at an even number?

4. (a) What is the probability of obtaining a 6 or a 1 on one roll of a cube with numbered faces?

(b) What is the probability of not getting a 6 or a 1 on one roll of a cube with numbered faces?

5. (a) In a bag there are 8 white balls and two red balls. What is the probability of not picking a red ball in one

draw?

6. (a) What is the sum of the probabilities in Problem 4 (a) and (b)? Can you interpret this as the probability of a certain event?
   (b) What is the sum of the probabilities in Problem 5 (a) and (b)? Can you interpret this as the probability of a certain event?
   (c) Could you use the information which you have obtained in parts (a) and (b) to solve Probelm 4(b) from the results in Problem 4(a) and to solve Problem 5(b) from the results in Problem 5(a)?
7. Let A be any event. Let B be the event "A does not occur". Write an equation which relates $P(A)$ and $P(B)$.
8. A bag contains 3 white balls and 2 red balls.
   (a) What is the probability of drawing a red ball on the first draw?
   (b) Suppose a white ball is drawn on the first attempt and not put back in the bag. What is the probability of now drawing a red ball on the next draw?

   Note: This is an example of sampling without replacements. After the first event happened, in this case a white ball was drawn, it was not replaced.

12-4 Independent Events

In the preceding discussion a formula was used to find probabilities when the events were mutually exclusive. This is one way to do such a problem. We will now study a formula for probability when one event has no effect upon another event, that is the two events are independent of each other.

Consider the tossing of a coin and the spinning of a pointer of a dial with 1, 2, 3 and 4, equally spaced. If the coin is tossed, no matter what side of the coin appears up, this outcome has no effect upon the outcome of the spinning pointer. This is an example of independence. If we let A be the event of a head coming up when the coin is tossed and B be the event of the pointer stopping at 4, then A and B are independent events.

If we wish to find the probability of a head appearing and the pointer stopping at 4 we are looking for the probability that both events will occur. If we let "A and B" stand for the event "both A and B occur" then we are looking for $P(A \text{ and } B)$.

By listing all possibilities we obtain the following:

$$\{H - 1, H - 2, H - 3, H - 4, T - 1, T - 2, T - 3, T - 4\} .$$

The desired event is $\{H - 4\}$, one of eight possible outcomes.

Thus, $$P(A \text{ and } B) = \frac{1}{8} .$$

We can also solve the problem by finding $P(A)$ and also $P(B)$.

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{4} .$$

Notice that $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ which is the probability that we found for event $(A \text{ and } B)$. In practice it is generally easier to use a formula for this type of probability problem.

Property. If event A and event B are two independent events,

then $$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example: You are taking a test of multiple choice questions where there are 5 choices of answers for each question. You have answered all the questions except Questions 7 and 9 which are troublesome. By elimination, you know that the correct answer for 7 is one of 2 selections, and the correct answer for 9 is one of 3 selections. You decide to guess. Find the probability of getting 7 and 9 correct.

Let A be the event that you choose the correct answer for Question 7, and B be the event that you choose the correct answer for Question 9. Events A and B are independent. (Why?) We are asking for P(A and B) . By the property we know that

$$P(A \text{ and } B) = P(A) \cdot P(B) .$$

We also know that $P(A) = \frac{1}{2}$. (Why? What does it mean to "guess"?) Also $P(B) = \frac{1}{3}$. Therefore

$$P(A \text{ and } B) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} .$$

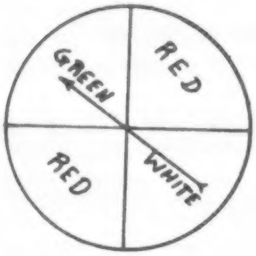
The probability of getting both Questions 7 and 9 correct by guesswork is $\frac{1}{6}$.

Exercises 12-4

1. You toss a coin twice in succession. Let A be the event of heads coming up on the first toss of the coin. Let B be the event of heads coming up on the second toss.
   - (a) Are events A and B independent? Explain.
   - (b) Use the appropriate property to find the probability that the coin will come up heads on both tosses.
2. (a) Can you state a property like the one in this section which holds for 3 independent events? Can you state a property which holds for four independent events? For any number of independent events?
   - (b) Find the probability of heads coming up each of nine successive tosses of a coin.
3. Which of the following pairs of events are independent?
   - (a) Picking a black ball in two draws from a bag containing black and while balls if you do not replace the first ball drawn.
   - (b) Picking a black ball in two draws from a bag containing black and white balls if you replace the first ball drawn.
   - (c) Going to school and becoming a lawyer.
   - (d) Throwing a 3 on a cube with numbered faces and tossing a head on a coin.
   - (e) The event of a day being sunny and the event of it being cloudy.

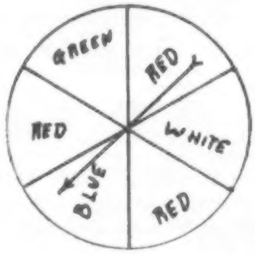
4. Your basket-ball team is to play team A and team B on two successive dates. It is estimated that the probability of winning over A is 1/2 and over B is 2/5.
   (a) What is the probability of your team winning both games?
   (b) If your team won the first game, what is the probability of winning the second?

5.



the four sections are equal.

Dial A



the six sections are equal.

Dial B

   (a) Both pointers are made to spin. Assuming both are honest, what is the probability that both will stop on red?
   (b) What is the probability that both will stop on green?
   (c) What is the probability that A stops on white and B stops on blue?

6. If you have a cube with numbered faces and a deck of cards what is the probability of throwing a four and drawing a spade?

7. If you have a bag of 5 black balls and 4 white balls, what is the chance of drawing 2 white balls from the bag if one is drawn and then replaced?

8. In problem 7, what is the chance of drawing two white balls if the first one is not replaced?

9. If a committee of 3 is to be chosen from a class of 20 children and each child is as likely to be chosen as any other child, what is the chance of you and your two best friends to be chosen?

10. A certain problem is to be solved. The chance that one man will solve the problem is 2/3. The chance that another man will solve the problem is 5/12. What is the chance that the problem not be solved when both men are independently working on it? What is the chance that it will be solved?

*11. When six coins are tossed, what is the chance that at least one coin will turn up a head?

12. BRAINBUSTER

Ten slips of paper numbered 1 to 10 are put in a hat and thoroughly mixed. Two slips of paper are drawn by a blindfolded person. What is the probability

(a) That the numbers are both even?
(b) That the sum of the two numbers is even?
(c) That the sum of the two numbers is divisible by 3?
(d) That the sum of the two numbers is less than 20?
(e) That the sum of the two numbers is more than 20?

13. BRAINBUSTER

From a deck of playing cards, five cards are drawn. What is the probability that

(a) four of the cards are aces?
(b) the hand contains an ace, a king, a queen, a jack and a ten?

14. BRAINBUSTER

(a) A penny, a nickle, a dime and a quarter are thrown and exactly two come up heads. What is the probability that one of those coming up a head is the dime?
(b) If the same four coins are thrown and exactly three come up heads, what is the probability that one of the three is the dime?

15. BRAINBUSTER

Five different coins are thrown (a half doller in addition to those above). What is the probability of each of the following?

(a) If exactly three come up heads one is a dime and one is a quarter.
(b) If exactly two come up heads one is a dime.
(c) That exactly two come up heads and one of these is the dime.
(d) That exactly three come up heads and two of these are a dime and a quarter.

16. BRAINBUSTER

There are ten sticks. One is an inch long, one is 2 inches long and so on up to ten inches long. A person picks up three of these sticks without looking. What is the probability that he can form a triangle with them?